Helsinki University of Technology
Mathematics

Somersalo/Bingham

Mat-1.198 Scattering Theory

4$^{\text{th}}$ set of exercises, 26.2.2003

1. Consider the interior Dirichlet problem

$$\begin{aligned} \left(\triangle + k^2\right) u &= 0 \qquad \text{in } D \\ u\big|_{\partial D} &= 0 \end{aligned} \tag{I.D.}$$

   where $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$. Find a condition for the resonances k, i.e., for those values of k for which (I.D.) has a non-trivial solution.

2. As Problem 1, but consider the Neumann condition,

$$\frac{\partial u}{\partial n}\bigg|_{\partial D} = 0.$$

   What is the smallest resonance? How do the resonances change when the radius of the disc changes?

3. By using a spherical harmonics expansion, solve the scattering problem

$$\begin{aligned} \left(\triangle + k^2\right) u &= 0 \qquad\qquad \text{in } \mathbb{R}^2 \setminus \overline{D} = \{x \in \mathbb{R}^2 \mid |x| > 1\} \\ u\big|_{\partial D} &= 0 \\ u &= u_{\text{inc}} + u_{\text{sc}} \end{aligned}$$

   where u_{sc} satisfies the radiation condition and

$$u_{\text{inc}}(x) = e^{ik\hat{\alpha}\cdot x}, \qquad |\hat{\alpha}| = 1.$$

   $\Big($Hint: Use $\frac{1}{2\pi}\int_0^{2\pi} e^{i(z\cos\theta + n\theta)}d\theta = i^n J_n(z)$.$\Big)$

4. The integral kernel of the single layer operator on the unit circle was

$$L(t,s) = L_1(t,s)\ln\left(4\sin^2\frac{t-s}{2}\right) + L_2(t,s),$$

   where

$$L_1(t,s) = -\frac{1}{4\pi}J_0\left(2k\sin\frac{t-s}{2}\right)$$

   and

$$L_2(t,s) = \frac{i}{4}H_0^{(1)}\left(2k\left|\sin\frac{t-s}{2}\right|\right) - L_1(t,s)\ln\left(4\sin^2\frac{t-s}{2}\right)$$

   is the regular part. We need to calculate $L_2(t,t)$. Calculate $L_2(t,t)$ by using the asymptotics of $H_0^{(1)}$ at the origin,

$$H_0^{(1)}(z) = J_0(z) + \frac{2i}{\pi}\left(\ln\frac{z}{2} + \gamma\right)J_0(z) + \sum_{k=1}^{\infty} a_k z^{2k}.$$