Helsinki University of Technology Somersalo/Bingham
Mathematics

Mat-1.198 Scattering Theory

2^{nd} set of exercises, 5.2.2003

1. Consider the acoustic medium scattering problem of solving

$$(\Delta + k^2 n(x))u(x) = 0 \text{ in } \mathbb{R}^3,$$

where $n(x) = c_0^2/c(x)^2 \in C(\mathbb{R}^3)$ is the refractive index of the medium, and $n = 1$ outside a bounded domain D. We write

$$u = u_{\rm inc} + u_{\rm sc},$$

where the *incoming field* $u_{\rm inc}$ is known and satsifies the Helmholtz equation in the whole $\mathbb{R}^3$, and the *scattered field* $u_{\rm sc}$ satisfies the Sommerfeld radiation condition at infinity. By using the Helmholtz representation formula, derive the *Lippmann-Schwinger equation* for u,

$$u(x) = u_{\rm inc}(x) + k^2 \int_D \Phi(x-y)(n(y)-1)u(y)dy.$$

2. Let the incoming field in the previous excercise be a plane wave propagating in the direction $\widehat{\alpha}$

$$u_{\rm inc}(x) = e^{ik\widehat{\alpha}\cdot x}, \quad \widehat{\alpha} \in S^2.$$

(a) Write the far field pattern $u_\infty(\widehat{x})$ for the scattered field.

(b) Consider the *Born approximation* of the solution,

$$u(x) \approx u_{\rm B}(x) = u_{\rm inc} + k^2 \int_D \Phi(x-y)(n(y)-1)u_{\rm inc}(y)dy.$$

What is the far field pattern of $u_{\rm B}$?

3. Calculate the far field pattern of the Born approximation explicitly, when the refractive index is given as

$$n(x) = \begin{cases} 1+h, & |x| < R \\ 1, & |x| \geq R \end{cases}$$

where $h > 0$ is a constant.

How does the approximate far field pattern behave in the (a) forward scattering direction, $\widehat{x} = \widehat{\alpha}$, and (b) the Backscattering direction, $\widehat{x} = -\widehat{\alpha}$?

4. Let $D \subset \mathbb{R}^3$ be a bounded domain containing the scatterer, so that outside D the field u satisfies the Helmholtz equation. As in the previous problems, let

$$u = u_{\rm inc} + u_{\rm sc},$$

where the incoming field satisfies the Helmholtz equation in the whole $\mathbb{R}^3$ and the scattered field satisifes the radiation condition. Show that then

$$u_{\rm sc}(x) = \int_{\partial D} \left(u(y) \frac{\partial \Phi}{\partial n(y)}(x-y) - \Phi(x-y) \frac{\partial u}{\partial n(y)}(y) \right) dS,$$

i.e., in the Helmholtz representation formula, we may replace $u_{\rm sc}$ by u.