8 Commutative Banach algebras

In this section we are interested in maximal ideals of commutative Banach algebras. We shall learn that such algebras are closely related to algebras of continuous functions on compact Hausdorff spaces: there is a natural far from trivial homomorphism from a commutative Banach algebra $\mathcal{A}$ to an algebra of functions on the set $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$, which can be endowed with a canonical topology — related mathematics is called the **Gelfand theory**. In the sequel, one should ponder this dilemma: which is more fundamental, a space or algebras of functions on it?

Examples of commutative Banach algebras:

1. Our familiar $C(K)$, when K is a compact space.
2. $L^\infty([0,1])$, when $[0,1]$ is endowed with the Lebesgue measure.
3. $A(\Omega) := C(\overline{\Omega}) \cap H(\Omega)$, when $\Omega \subset \mathbb{C}$ is open and $\overline{\Omega} \subset \mathbb{C}$ is compact.
4. $M(\mathbb{R}^n)$, the convolution algebra of complex Borel measures on $\mathbb{R}^n$, with the Dirac delta distribution at $0 \in \mathbb{R}^n$ as the unit element, and endowed with the total variation norm.
5. The algebra of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}$; notice that this algebra contains nilpotent elements!

Spectrum of algebra. The *spectrum of an algebra* $\mathcal{A}$ is

$$\mathrm{Spec}(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}, \mathbb{C}),$$

i.e. the set of homomorphisms $\mathcal{A} \to \mathbb{C}$; such a homomorphism is called a *character of* $\mathcal{A}$.

Remark. The concept of spectrum is best suited for **commutative** algebras, as $\mathbb{C}$ is a commutative algebra; here a character $\mathcal{A} \to \mathbb{C}$ should actually be considered as an algebra representation $\mathcal{A} \to \mathcal{L}(\mathbb{C})$. In order to fully capture the structure of a **non-commutative** algebra, we should study representations of type $\mathcal{A} \to \mathcal{L}(X)$, where the vector spaces X are multi-dimensional; for instance, if $\mathcal{H}$ is a Hilbert space of dimension 2 or greater then $\mathrm{Spec}(\mathcal{L}(\mathcal{H})) = \emptyset$. However, the spectrum of a commutative Banach algebra is rich, as there is a bijective correspondence between characters and maximal ideals. Moreover, the spectrum of the algebra is akin to the spectra of its elements:

Theorem (Gelfand, 1940). *Let $\mathcal{A}$ be a commutative Banach algebra. Then:*
(a) *Every maximal ideal of $\mathcal{A}$ is of the form* $\mathrm{Ker}(h)$ *for some* $h \in \mathrm{Spec}(\mathcal{A})$*;*
(b) $\mathrm{Ker}(h)$ *is a maximal ideal for every* $h \in \mathrm{Spec}(\mathcal{A})$*;*
(c) $x \in \mathcal{A}$ *is invertible if and only if* $\forall h \in \mathrm{Spec}(\mathcal{A}): \ h(x) \neq 0$*;*
(d) $x \in \mathcal{A}$ *is invertible if and only if it is not in any ideal of* $\mathcal{A}$*;*
(e) $\sigma(x) = \{h(x) \mid h \in \ \mathrm{Spec}(\mathcal{A})\}$.

Proof.

(a) Let $\mathcal{M} \subset \mathcal{A}$ be a maximal ideal; let $[x] := x + \mathcal{M}$ for $x \in \mathcal{A}$. Since $\mathcal{A}$ is commutative and $\mathcal{M}$ is maximal, every non-zero element in the quotient algebra $\mathcal{A}/\mathcal{M}$ is invertible. We know that $\mathcal{M}$ is closed, so that $\mathcal{A}/\mathcal{M}$ is a Banach algebra. Due to the Gelfand–Mazur Theorem, there exists an isometric isomorphism $\lambda \in \mathrm{Hom}(\mathcal{A}/\mathcal{M}, \mathbb{C})$. Then

$$h = (x \mapsto \lambda([x])) : \mathcal{A} \to \mathbb{C}$$

is a character, and

$$\mathrm{Ker}(h) = \mathrm{Ker}((x \mapsto [x]) : \mathcal{A} \to \mathcal{A}/\mathcal{M}) = \mathcal{M}.$$

(b) Let $h : \mathcal{A} \to \mathbb{C}$ be a character. Now h is a linear mapping, so that the co-dimension of $\mathrm{Ker}(h)$ in $\mathcal{A}$ equals the dimension of $h(\mathcal{A}) \subset \mathbb{C}$, which clearly is 1. Any ideal of co-dimension 1 in an algebra must be maximal, so that $\mathrm{Ker}(h)$ is maximal.

(c) If $x \in \mathcal{A}$ is invertible and $h \in \mathrm{Spec}(\mathcal{A})$ then $h(x) \in \mathbb{C}$ is invertible, that is $h(x) \neq 0$. For the converse, assume that $x \in \mathcal{A}$ is non-invertible. Then

$$\mathcal{A}x = \{ax \mid a \in \mathcal{A}\}$$

is an ideal of $\mathcal{A}$ (notice that $\mathbb{I} = ax = xa$ would mean that $a = x^{-1}$). Hence by Krull's Theorem, there is a maximal ideal $\mathcal{M} \subset \mathcal{A}$ such that $\mathcal{A}x \subset \mathcal{M}$. Then (a) provides a character $h \in \mathrm{Spec}(\mathcal{A})$ for which $\mathrm{Ker}(h) = \mathcal{M}$. Especially, $h(x) = 0$.

(d) This follows from (a,b,c) directly.

(e) (c) is equivalent to
"$x \in \mathcal{A}$ non-invertible if and only if $\exists h \in \mathrm{Spec}(\mathcal{A}): \ h(x) = 0$",
which is equivalent to
"$\lambda\mathbb{I} - x$ non-invertible if and only if $\exists h \in \mathrm{Spec}(\mathcal{A}): \ h(x) = \lambda$" $\square$

Exercise. Let $\mathcal{A}$ be a Banach algebra and $x, y \in \mathcal{A}$ such that $xy = yx$. Prove that $\sigma(x+y) \subset \sigma(x) + \sigma(y)$ and $\sigma(xy) \subset \sigma(x)\sigma(y)$.

Exercise. Let $\mathcal{A}$ be the algebra of those functions $f : \mathbb{R} \to \mathbb{C}$ for which

$$f(x) = \sum_{n \in \mathbb{Z}} f_n \, e^{ix \cdot n}, \quad \|f\| = \sum_{n \in \mathbb{Z}} |f_n| < \infty.$$

Show that $\mathcal{A}$ is a commutative Banach algebra. Show that if $f \in \mathcal{A}$ and $\forall x \in \mathbb{R} : \ f(x) \neq 0$ then $1/f \in \mathcal{A}$.

Gelfand transform. Let $\mathcal{A}$ be a commutative Banach algebra. The *Gelfand transform* $\widehat{x}$ *of an element* $x \in \mathcal{A}$ is the function

$$\widehat{x} : \mathrm{Spec}(\mathcal{A}) \to \mathbb{C}, \quad \widehat{x}(\phi) := \phi(x).$$

Let $\widehat{\mathcal{A}} := \{\widehat{x} : \mathrm{Spec}(\mathcal{A}) \to \mathbb{C} \mid x \in \mathcal{A}\}$. The mapping

$$\mathcal{A} \to \widehat{\mathcal{A}}, \quad x \mapsto \widehat{x},$$

is called the *Gelfand transform of* $\mathcal{A}$. We endow the set $\mathrm{Spec}(\mathcal{A})$ with the $\widehat{\mathcal{A}}$-induced topology, called the *Gelfand topology*; this topological space is called the *maximal ideal space of* $\mathcal{A}$ (for a good reason, in the light of the previous theorem). In other words, the Gelfand topology is the weakest topology on $\mathrm{Spec}(\mathcal{A})$ making every $\widehat{x}$ a continuous function, i.e. the weakest topology on $\mathrm{Spec}(\mathcal{A})$ for which $\widehat{\mathcal{A}} \subset C(\mathrm{Spec}(\mathcal{A}))$.

Theorem (Gelfand, 1940). *Let* $\mathcal{A}$ *be a commutative Banach algebra. Then* $K = \mathrm{Spec}(\mathcal{A})$ *is a compact Hausdorff space in the Gelfand topology, the Gelfand transform is a continuous homomorphism* $\mathcal{A} \to C(K)$*, and* $\|\widehat{x}\| = \sup_{\phi \in K} |\widehat{x}(\phi)| = \rho(x)$ *for every* $x \in \mathcal{A}$.

Proof. The Gelfand transform is a homomorphism, since

$$\begin{aligned}
\widehat{\lambda x}(\phi) = \phi(\lambda x) = \lambda \phi(x) = \lambda \widehat{x}(\phi) &= (\lambda \widehat{x})(\phi), \\
\widehat{x+y}(\phi) = \phi(x+y) = \phi(x) + \phi(y) = \widehat{x}(\phi) + \widehat{y}(\phi) &= (\widehat{x} + \widehat{y})(\phi), \\
\widehat{xy}(\phi) = \phi(xy) = \phi(x)\phi(y) = \widehat{x}(\phi)\widehat{y}(\phi) &= (\widehat{x}\widehat{y})(\phi), \\
\widehat{\mathbb{I}_{\mathcal{A}}}(\phi) = \phi(\mathbb{I}_{\mathcal{A}}) = 1 &= \mathbb{I}_{C(K)}(\phi),
\end{aligned}$$

for every $\lambda \in \mathbb{C}$, $x, y \in \mathcal{A}$ and $\phi \in K$. Moreover,

$$\widehat{x}(K) = \{\widehat{x}(\phi) \mid \phi \in K\} = \{\phi(x) \mid \phi \in \mathrm{Spec}(\mathcal{A})\} = \sigma(x),$$

implying

$$\|\widehat{x}\| = \rho(x) \leq \|x\|.$$

Clearly K is a Hausdorff space. What about compactness? Now $K = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ is a subset of the closed unit ball of the dual Banach space $\mathcal{A}'$; by the Banach–Alaoglu Theorem, this unit ball is compact in the weak*-topology. Recall that the weak*-topology $\tau_{\mathcal{A}'}$ of $\mathcal{A}'$ is the $\mathcal{A}$-induced topology, with the interpretation $\mathcal{A} \subset \mathcal{A}''$; thus the Gelfand topology τ_K is the relative weak*-topology, i.e.

$$\tau_K = \tau_{\mathcal{A}'}|_K.$$

To prove that τ_K is compact, it is sufficient to show that $K \subset \mathcal{A}'$ is closed in the weak*-topology.

Let $f \in \mathcal{A}'$ be in the weak*-closure of K. We have to prove that $f \in K$, i.e.

$$f(xy) = f(x)f(y) \quad \text{and} \quad f(\mathbb{I}) = 1.$$

Let $x, y \in \mathcal{A}$, $\varepsilon > 0$. Let $S := \{\mathbb{I}, x, y, xy\}$. Using the notation of the proof of Banach–Alaoglu Theorem,

$$U(f, S, \varepsilon) = \{\psi \in \mathcal{A}' : \ z \in S \Rightarrow |\psi z - f z| < \varepsilon\}$$

is a weak*-neighborhood of f. Thus choose $h_\varepsilon \in K \cap U(f, S, \varepsilon)$. Then

$$|1 - f(\mathbb{I})| = |h_\varepsilon(\mathbb{I}) - f(\mathbb{I})| < \varepsilon;$$

$\varepsilon > 0$ being arbitrary, we have $f(\mathbb{I}) = 1$. Noticing that $|h_\varepsilon(x)| \leq \|x\|$, we get

$$\begin{aligned}
& |f(xy) - f(x)f(y)| \\
\leq \ & |f(xy) - h_\varepsilon(xy)| + |h_\varepsilon(xy) - h_\varepsilon(x)f(y)| + |h_\varepsilon(x)f(y) - f(x)f(y)| \\
= \ & |f(xy) - h_\varepsilon(xy)| + |h_\varepsilon(x)| \cdot |h_\varepsilon(y) - f(y)| + |h_\varepsilon(x) - f(x)| \cdot |f(y)| \\
\leq \ & \varepsilon \ (1 + \|x\| + |f(y)|).
\end{aligned}$$

This holds for every $\varepsilon > 0$, so that actually

$$f(xy) = f(x)f(y);$$

we have proven that f is a homomorphism, $f \in K$ $\square$

Exercise. Let $\mathcal{A}$ be a commutative Banach algebra. Its *radical* $\mathrm{Rad}(\mathcal{A})$ is the intersection of all the maximal ideals of $\mathcal{A}$. Show that

$$\mathrm{Rad}(\mathcal{A}) = \mathrm{Ker}(x \mapsto \widehat{x}) = \{x \in \ \mathcal{A} \mid \rho(x) = 0\},$$

where $x \mapsto \widehat{x}$ is the Gelfand transform. Show that nilpotent elements of $\mathcal{A}$ belong to the radical.

Exercise. Let X be a finite set. Describe the Gelfand transform of $\mathcal{F}(X)$.

Exercise. Describe the Gelfand transform of the algebra of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$, where $\alpha, \beta \in \mathbb{C}$.

Theorem. *Let X be a compact Hausdorff space. Then* $\mathrm{Spec}(C(X))$ *is homeomorphic to X.*

Proof. For $x \in X$, let us define the function

$$h_x : C(X) \to \mathbb{C}, \quad f \mapsto f(x) \quad (\text{evaluation at } x \in X).$$

This is clearly a homomorphism, and hence we may define the mapping

$$\phi : X \to \mathrm{Spec}(C(X)), \quad x \mapsto h_x.$$

Let us prove that ϕ is a homeomorphism.

If $x, y \in X$, $x \neq y$, then Urysohn's Lemma provides $f \in C(X)$ such that $f(x) \neq f(y)$. Thereby $h_x(f) \neq h_y(f)$, yielding $\phi(x) = h_x \neq h_y = \phi(y)$; thus ϕ is injective. It is also surjective: Namely, let us **assume** that $h \in \mathrm{Spec}(C(X)) \setminus \phi(X)$. Now $\mathrm{Ker}(h) \subset C(X)$ is a maximal ideal, and for every $x \in X$ we may choose

$$f_x \in \mathrm{Ker}(h) \setminus \mathrm{Ker}(h_x) \subset C(X).$$

Then $U_x := f_x^{-1}(\mathbb{C} \setminus \{0\}) \in \mathcal{V}(x)$, so that

$$\mathcal{U} = \{U_x \mid x \in X\}$$

is an open cover of X, which due to the compactness has a finite subcover $\{U_{x_j}\}_{j=1}^n \subset \mathcal{U}$. Since $f_{x_j} \in \mathrm{Ker}(h)$, the function

$$f := \sum_{j=1}^n |f_{x_j}|^2 = \sum_{j=1}^n f_{x_j} \overline{f_{x_j}}$$

belongs to $\mathrm{Ker}(h)$. Clearly $f(x) \neq 0$ for every $x \in X$. Therefore $g \in C(X)$ with $g(x) = 1/f(x)$ is the inverse element of f; this is a **contradiction**, since no invertible element belongs to an ideal. Thus ϕ must be surjective.

We have proven that $\phi : X \to \mathrm{Spec}(C(X))$ is a bijection. Thereby X and $\mathrm{Spec}(C(X))$ can be identified as sets. The Gelfand-topology of $\mathrm{Spec}(C(X))$ is then identified with the $C(X)$-induced topology σ of X, which is weaker than the original topology τ of X. Hence $\phi : (X, \tau) \to \mathrm{Spec}(C(X))$ is continuous. Actually, $\sigma = \tau$, because a continuous bijection from a compact space to a Hausdorff space is a homeomorphism $\square$

Corollary. *Let X and Y be compact Hausdorff spaces. Then the Banach algebras $C(X)$ and $C(Y)$ are isomorphic if and only if X is homeomorphic to Y.*

Proof. By the previous Theorem, $X \cong \mathrm{Spec}(C(X))$ and $Y \cong \mathrm{Spec}(C(Y))$. If $C(X)$ and $C(Y)$ are isomorphic Banach algebras then

$$X \cong \mathrm{Spec}(C(X)) \overset{C(X)\cong C(Y)}{\cong} \mathrm{Spec}(C(Y)) \cong Y.$$

Conversely, a homeomorphism $\phi : X \to Y$ begets a Banach algebra isomorphism

$$\Phi : C(Y) \to C(X), \quad (\Phi f)(x) := f(\phi(x)),$$

as the reader easily verifies $\square$