Miscellany on (Banach) algebras

Exercise. Let $\{\mathcal{A}_j \mid j \in J\}$ be a family of topological algebras. Endow $\mathcal{A} := \prod_{j \in J} \mathcal{A}_j$ with a structure of a topological algebra.

Before examining commutative Banach algebras in detail, we derive some useful results that could have been treated already in earlier sections (but we forgot to do so :)

Lemma. *Let $\mathcal{A}$ be a commutative algebra and $\mathcal{M}$ be its ideal. Then $\mathcal{M}$ is maximal if and only if $[0]$ is the only non-invertible element of $\mathcal{A}/\mathcal{M}$.*

Proof. Of course, here $[x]$ means $x + \mathcal{M}$, where $x \in \mathcal{A}$. Assume that $\mathcal{M}$ is a maximal ideal. Take $[x] \neq [0]$, so that $x \notin \mathcal{M}$. Define

$$\mathcal{J} := \mathcal{A}x + \mathcal{M} = \{ax + m \mid a \in \mathcal{A},\ m \in \mathcal{M}\}.$$

Then clearly $\mathcal{J} \neq \mathcal{M} \subset \mathcal{J}$, and $\mathcal{J}$ is a vector subspace of $\mathcal{A}$. If $y \in \mathcal{A}$ then

$$\mathcal{J}y = y\mathcal{J} = y\mathcal{A}x + y\mathcal{M} \subset \mathcal{A}x + \mathcal{M} = \mathcal{J},$$

so that either $\mathcal{J}$ is an ideal or $\mathcal{J} = \mathcal{A}$. But since $\mathcal{M}$ is a maximal ideal contained properly in $\mathcal{J}$, we must have $\mathcal{J} = \mathcal{A}$. Thus there exist $a \in \mathcal{A}$ and $m \in \mathcal{M}$ such that $ax + m = \mathbb{I}_{\mathcal{A}}$. Then

$$[a][x] = \mathbb{I}_{\mathcal{A}/\mathcal{M}} = [x][a],$$

$[x]$ is invertible in $\mathcal{A}/\mathcal{M}$.

Conversely, assume that all the non-zero elements are invertible in $\mathcal{A}/\mathcal{M}$. Assume that $\mathcal{J} \subset \mathcal{A}$ is an ideal containing $\mathcal{M}$. **Suppose** $\mathcal{J} \neq \mathcal{M}$, and pick $x \in \mathcal{J} \setminus \mathcal{M}$. Now $[x] \neq [0]$, so that for some $y \in \mathcal{A}$ we have $[x][y] = [\mathbb{I}_{\mathcal{A}}]$. Thereby

$$\mathbb{I}_{\mathcal{A}} \in xy + \mathcal{M} \overset{x \in \mathcal{J}}{\subset} \mathcal{J} + \mathcal{M} \subset \mathcal{J} + \mathcal{J} = \mathcal{J},$$

which is a **contradiction**, since no ideal can contain invertible elements. Therefore we must have $\mathcal{J} = \mathcal{M}$, meaning that $\mathcal{M}$ is maximal □

Proposition. *A maximal ideal in a Banach algebra is closed.*

Proof. In a topological algebra, the closure of an ideal is either an ideal or the whole algebra. Let $\mathcal{M}$ be a maximal ideal of a Banach algebra $\mathcal{A}$. The set $G(\mathcal{A}) \subset \mathcal{A}$ of the invertible elements is open, and $\mathcal{M} \cap G(\mathcal{A}) = \emptyset$ (because no ideal contains invertible elements). Thus $\mathcal{M} \subset \overline{\mathcal{M}} \subset \mathcal{A} \setminus G(\mathcal{A})$, so that $\overline{\mathcal{M}}$ is an ideal containing a maximal ideal $\mathcal{M}$; thus $\overline{\mathcal{M}} = \mathcal{M}$ □

Proposition. *Let $\mathcal{J}$ be a closed ideal of a Banach algebra $\mathcal{A}$. Then the quotient vector space $\mathcal{A}/\mathcal{J}$ is a Banach algebra; moreover, $\mathcal{A}/\mathcal{J}$ is commutative if $\mathcal{A}$ is commutative.*

Proof. Let us denote $[x] := x + \mathcal{J}$ for $x \in \mathcal{A}$. Since $\mathcal{J}$ is a closed vector subspace, the quotient space $\mathcal{A}/\mathcal{J}$ is a Banach space with the norm

$$[x] \mapsto \|[x]\| = \inf_{j \in \mathcal{J}} \|x + j\|.$$

Let $x, y \in \mathcal{A}$ and $\varepsilon > 0$. Then there exist $i, j \in \mathcal{J}$ such that

$$\|x + i\| \leq \|[x]\| + \varepsilon, \quad \|y + j\| \leq \|[y]\| + \varepsilon.$$

Now $(x + i)(y + j) \in [xy]$, so that

$$\begin{aligned} \|[xy]\| &\leq \|(x+i)(y+j)\| \\ &\leq \|x+i\| \; \|y+j\| \\ &\leq (\|[x]\| + \varepsilon) \; (\|[y]\| + \varepsilon) \\ &= \|[x]\| \; \|[y]\| + \varepsilon(\|[x]\| + \|[y]\| + \varepsilon); \end{aligned}$$

since $\varepsilon > 0$ is arbitrary, we have

$$\|[x][y]\| \leq \|[x]\| \; \|[y]\|.$$

Finally, $\|[\mathbb{I}]\| \leq \|\mathbb{I}\| = 1$ and $\|[x]\| = \|[x][\mathbb{I}]\| \leq \|[x]\| \; \|[\mathbb{I}]\|$, so that we have $\|[\mathbb{I}]\| = 1$ □

Exercise*. Let $\mathcal{A}$ be an algebra. The *commutant* of a subset $\mathcal{S} \subset \mathcal{A}$ is

$$\Gamma(\mathcal{S}) := \{x \in \mathcal{A} \mid \forall y \in \mathcal{S} : \; xy = yx\}.$$

Prove the following claims:

(a) $\Gamma(\mathcal{S}) \subset \mathcal{A}$ is a subalgebra; $\Gamma(\mathcal{S})$ is closed if $\mathcal{A}$ is a topological algebra.

(b) $\mathcal{S} \subset \Gamma(\Gamma(\mathcal{S}))$.

(c) If $xy = yx$ for every $x, y \in \mathcal{S}$ then $\Gamma(\Gamma(\mathcal{S})) \subset \mathcal{A}$ is a commutative subalgebra, where $\sigma_{\Gamma(\Gamma(\mathcal{S}))}(z) = \sigma_{\mathcal{A}}(z)$ for every $z \in \Gamma(\Gamma(\mathcal{S}))$.