6 Banach algebras

Banach algebra. An algebra $\mathcal{A}$ is called a *Banach algebra* if it is a Banach space satisfying

$$\|xy\| \leq \|x\| \, \|y\|$$

for every $x, y \in \mathcal{A}$ and

$$\|\mathbb{I}\| = 1.$$

The next exercise is very important:

Exercise*. *Let K be a compact space. Show that $C(K)$ is a Banach algebra with the norm $f \mapsto \|f\| = \max_{x \in K} |f(x)|$.*

Examples. Let X be a Banach space. Then the Banach space $\mathcal{L}(X)$ of bounded linear operators $X \to X$ is a Banach algebra, when the multiplication is the composition of operators, since

$$\|AB\| \leq \|A\| \, \|B\|$$

for every $A, B \in \mathcal{L}(X)$; the unit is the identity operator $I : X \to X$, $x \mapsto x$. Actually, this is not far away from characterizing all the Banach algebras:

Theorem. *A Banach algebra $\mathcal{A}$ is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(X)$ for a Banach space X.*

Proof. Here $X := \mathcal{A}$. For $x \in \mathcal{A}$, let us define

$$m(x) : \mathcal{A} \to \mathcal{A} \quad \text{by} \quad m(x)y := xy.$$

Obviously $m(x)$ is a linear mapping, $m(xy) = m(x)m(y)$, $m(\mathbb{I}_\mathcal{A}) = \mathbb{I}_{\mathcal{L}(\mathcal{A})}$, and

$$\begin{aligned} \|m(x)\| &= \sup_{y \in \mathcal{A}:\ \|y\| \leq 1} \|xy\| \\ &\leq \sup_{y \in \mathcal{A}:\ \|y\| \leq 1} (\|x\| \, \|y\|) = \|x\| = \|m(x)\mathbb{I}_\mathcal{A}\| \\ &\leq \|m(x)\| \, \|\mathbb{I}_\mathcal{A}\| = \|m(x)\|; \end{aligned}$$

briefly, $m = (x \mapsto m(x)) \in \mathrm{Hom}(\mathcal{A}, \mathcal{L}(\mathcal{A}))$ is isometric. Thereby $m(\mathcal{A}) \subset \mathcal{L}(\mathcal{A})$ is a closed subspace and a Banach algebra $\square$

Exercise*. Let a Banach space $\mathcal{A}$ be a topological algebra. Equip $\mathcal{A}$ with an equivalent Banach algebra norm.

Exercise. Let $\mathcal{A}$ be a Banach algebra, and let $x, y \in \mathcal{A}$ satisfy

$$x^2 = x, \quad y^2 = y, \quad xy = yx.$$

Show that either $x = y$ or $\|x - y\| \geq 1$. Give an example of a Banach algebra $\mathcal{A}$ with elements $x, y \in \mathcal{A}$ such that $x^2 = x \neq y = y^2$ and $\|x - y\| < 1$.

Proposition. *Let $\mathcal{A}$ be a Banach algebra. Then* $\mathrm{Hom}(\mathcal{A}, \mathbb{C}) \subset \mathcal{A}'$ *and* $\|\phi\| = 1$ *for every* $\phi \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$.

Proof. Let $x \in \mathcal{A}$, $\|x\| < 1$. Let

$$y_n := \sum_{j=0}^{n} x^j,$$

where $x^0 := \mathbb{I}$. If $n > m$ then

$$\begin{aligned} \|y_n - y_m\| &= \|x^m + x^{m+1} + \ldots + x^n\| \\ &\leq \|x\|^m \left(1 + \|x\| + \ldots + \|x\|^{n-m}\right) \\ &= \|x\|^m \, \frac{1 - \|x\|^{n-m+1}}{1 - \|x\|} \to_{n>m\to\infty} 0; \end{aligned}$$

thus $(y_n)_{n=1}^{\infty} \subset \mathcal{A}$ is a Cauchy sequence. There exists $y = \lim_{n\to\infty} y_n \in \mathcal{A}$, because $\mathcal{A}$ is complete. Since $x^n \to 0$ and

$$y_n(\mathbb{I} - x) = \mathbb{I} - x^{n+1} = (\mathbb{I} - x)y_n,$$

we deduce $y = (\mathbb{I} - x)^{-1}$. Suppose $\lambda = \phi(x)$, $|\lambda| > \|x\|$; now $\|\lambda^{-1}x\| = |\lambda|^{-1} \, \|x\| < 1$, so that $\mathbb{I} - \lambda^{-1}x$ is invertible. Then

$$\begin{aligned} 1 &= \phi(\mathbb{I}) = \phi\left((\mathbb{I} - \lambda^{-1}x)(\mathbb{I} - \lambda^{-1}x)^{-1}\right) \\ &= \phi\left(\mathbb{I} - \lambda^{-1}x\right) \, \phi\left((\mathbb{I} - \lambda^{-1}x)^{-1}\right) \\ &= (1 - \lambda^{-1}\phi(x)) \, \phi\left((\mathbb{I} - \lambda^{-1}x)^{-1}\right) = 0, \end{aligned}$$

a contradiction; hence

$$\forall x \in \mathcal{A} : \; |\phi(x)| \leq \|x\|,$$

that is $\|\phi\| \leq 1$. Finally, $\phi(\mathbb{I}) = 1$, so that $\|\phi\| = 1$ $\square$

Lemma. *Let $\mathcal{A}$ be a Banach algebra. The set $G(\mathcal{A}) \subset \mathcal{A}$ of its invertible elements is open. The mapping $(x \mapsto x^{-1}) : G(\mathcal{A}) \to G(\mathcal{A})$ is a homeomorphism.*

Proof. Take $x \in G(\mathcal{A})$ and $h \in \mathcal{A}$. As in the proof of the previous Proposition, we see that

$$x - h = x(\mathbb{I} - x^{-1}h)$$

is invertible if $\|x^{-1}\| \, \|h\| < 1$, that is $\|h\| < \|x^{-1}\|^{-1}$; thus $G(\mathcal{A}) \subset \mathcal{A}$ is open.

The mapping $x \mapsto x^{-1}$ is clearly its own inverse. Moreover

$$\begin{aligned}
\|(x-h)^{-1} - x^{-1}\| &= \|(\mathbb{I} - x^{-1}h)^{-1}x^{-1} - x^{-1}\| \\
&\le \|(\mathbb{I} - x^{-1}h)^{-1} - \mathbb{I}\| \, \|x^{-1}\| = \|\sum_{n=1}^{\infty}(x^{-1}h)^n\| \, \|x^{-1}\| \\
&\le \|h\| \left(\sum_{n=1}^{\infty} \|x^{-1}\|^{n+1} \, \|h\|^{n-1} \right) \to_{h \to 0} 0;
\end{aligned}$$

hence $x \mapsto x^{-1}$ is a homeomorphism □

Exercise. Let $\mathcal{A}$ be a Banach algebra. We say that $x \in \mathcal{A}$ is a *topological zero divisor* if there exists a sequence $(y_n)_{n=1}^{\infty} \subset \mathcal{A}$ such that $\|y_n\| = 1$ for all n and

$$\lim_{n\to\infty} x y_n = 0 = \lim_{n\to\infty} y_n x.$$

(a) Show that if $(x_n)_{n=1}^{\infty} \subset G(\mathcal{A})$ satisfies $x_n \to x \in \partial G(\mathcal{A})$ then $\|x_n^{-1}\| \to \infty$.
(b) Using this result, show that the boundary points of $G(\mathcal{A})$ are topological zero divisors.
(c) In what kind of Banach algebras 0 is the only topological zero divisor?

Theorem (Gelfand, 1939). *Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. The spectrum $\sigma(x) \subset \mathbb{C}$ is a non-empty compact set.*

Proof. Let $x \in \mathcal{A}$. Then $\sigma(x)$ belongs to a 0-centered disc of radius $\|x\|$ in the complex plane: for if $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$ then $\mathbb{I} - \lambda^{-1}x$ is invertible, equivalently $\lambda\mathbb{I} - x$ is invertible.

The mapping $g : \mathbb{C} \to \mathcal{A}$, $\lambda \mapsto \lambda\mathbb{I} - x$, is continuous; the set $G(\mathcal{A}) \subset \mathcal{A}$ of invertible elements is open, so that

$$\mathbb{C} \setminus \sigma(x) = g^{-1}(G(\mathcal{A}))$$

is open. Thus $\sigma(x) \in \mathbb{C}$ is closed and bounded, i.e. compact by Heine–Borel.

The hard part is to prove the non-emptiness of the spectrum. Let us define the *resolvent mapping* $R : \mathbb{C} \setminus \sigma(x) \to G(\mathcal{A})$ by

$$R(\lambda) = (\lambda \mathbb{I} - x)^{-1}.$$

We know that this mapping is continuous, because it is composed of continuous mappings

$$(\lambda \mapsto \lambda\mathbb{I} - x) : \mathbb{C} \setminus \sigma(x) \to G(\mathcal{A}) \quad \text{and} \quad (y \mapsto y^{-1}) : G(\mathcal{A}) \to G(\mathcal{A}).$$

We want to show that R is *weakly holomorphic*, that is $f \circ R \in H(\mathbb{C} \setminus \sigma(x))$ for every $f \in \mathcal{A}' = \mathcal{L}(\mathcal{A}, \mathbb{C})$. Let $z \in \mathbb{C} \setminus \sigma(x)$, $f \in \mathcal{A}'$. Then we calculate

$$\begin{aligned}
\frac{(f \circ R)(z+h) - (f \circ R)(z)}{h} &= f\left(\frac{R(z+h) - R(z)}{h}\right) \\
&= f\left(\frac{R(z+h)R(z)^{-1} - \mathbb{I}}{h} R(z)\right) \\
&= f\left(\frac{R(z+h)(R(z+h)^{-1} - h\mathbb{I}) - \mathbb{I}}{h} R(z)\right) \\
&= f(-R(z+h)R(z)) \\
&\to_{h \to 0} f(-R(z)^2),
\end{aligned}$$

because f and R are continuous; thus R is weakly holomorphic.

Suppose $|\lambda| > \|x\|$. Then

$$\begin{aligned}
\|R(\lambda)\| &= \|(\lambda\mathbb{I} - x)^{-1}\| = |\lambda|^{-1} \, \|(\mathbb{I} - x/\lambda)^{-1}\| = |\lambda|^{-1} \left\| \sum_{j=0}^{\infty} (x/\lambda)^j \right\| \\
&\leq |\lambda|^{-1} \sum_{j=0}^{\infty} \|x/\lambda\|^{-j} = |\lambda|^{-1} \frac{1}{1 - \|x/\lambda\|} = \frac{1}{|\lambda| - \|x\|} \\
&\to_{|\lambda| \to \infty} 0.
\end{aligned}$$

Thereby

$$(f \circ R)(\lambda) \to_{|\lambda| \to \infty} 0$$

for every $f \in \mathcal{A}'$. To get a contradiction, suppose $\sigma(x) = \emptyset$. Then $f \circ R \in H(\mathbb{C})$ is 0 by Liouville's Theorem (see Appendix), for every $f \in \mathcal{A}'$; the Hahn-Banach Theorem says that then $R(\lambda) = 0$ for every $\lambda \in \mathbb{C}$; this is a contradiction, since $0 \notin G(\mathcal{A})$. Thus $\sigma(x) \neq \emptyset$ $\square$

Exercise. Let $\mathcal{A}$ be a Banach algebra, $x \in \mathcal{A}$, $\Omega \subset \mathbb{C}$ an open set, and $\sigma(x) \subset \Omega$. Then

$$\exists \delta > 0 \ \forall y \in \mathcal{A} : \ \|y\| < \delta \Rightarrow \sigma(x+y) \subset \Omega.$$

Corollary (Gelfand–Mazur). *Let $\mathcal{A}$ be a Banach algebra where $0 \in \mathcal{A}$ is the only non-invertible element. Then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. Take $x \in \mathcal{A}$, $x \neq 0$. Since $\sigma(x) \neq \emptyset$, pick $\lambda(x) \in \sigma(x)$. Then $\lambda(x)\mathbb{I} - x$ is non-invertible, so that it must be 0; $x = \lambda(x)\mathbb{I}$. By defining $\lambda(0) = 0$, we have an algebra isomorphism

$$\lambda : \mathcal{A} \to \mathbb{C}.$$

Moreover, $|\lambda(x)| = \|\lambda(x)\mathbb{I}\| = \|x\|$ $\square$

Exercise. Let $\mathcal{A}$ be a Banach algebra, and suppose that there exists $C < \infty$ such that

$$\|x\| \ \|y\| \leq C \ \|xy\|$$

for every $x, y \in \mathcal{A}$. Show that $\mathcal{A} \cong \mathbb{C}$ isometrically.

Spectral radius. Let $\mathcal{A}$ be a Banach algebra. The *spectral radius* of $x \in \mathcal{A}$ is

$$\rho(x) := \sup_{\lambda \in \sigma(x)} |\lambda|;$$

this is well-defined, because due to Gelfand the spectrum in non-empty. In other words, $\overline{\mathbb{D}(0, \rho(x))} = \{\lambda \in \mathbb{C} : \ |\lambda| \leq \rho(x)\}$ is the smallest 0-centered closed disk containing $\sigma(x) \subset \mathbb{C}$. Notice that $\rho(x) \leq \|x\|$, since $\lambda\mathbb{I} - x = \lambda(\mathbb{I} - x/\lambda)$ is invertible if $|\lambda| > \|x\|$.

Spectral Radius Formula (Beurling, 1938; Gelfand, 1939). *Let $\mathcal{A}$ be a Banach algebra, $x \in \mathcal{A}$. Then*

$$\rho(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$

Proof. For $x = 0$ the claim is trivial, so let us assume that $x \neq 0$. By Gelfand's Theorem, $\sigma(x) \neq \emptyset$. Let $\lambda \in \sigma(x)$ and $n \geq 1$. Notice that in an algebra, if both ab and ba are invertible then the elements a, b are invertible. Therefore

$$\lambda^n \mathbb{I} - x^n = (\lambda \mathbb{I} - x) \left(\sum_{k=0}^{n-1} \lambda^{n-1-k} x^k \right) = \left(\sum_{k=0}^{n-1} \lambda^{n-1-k} x^k \right) (\lambda \mathbb{I} - x)$$

implies that $\lambda^n \in \sigma(x^n)$. Thus $|\lambda^n| \leq \|x^n\|$, so that

$$\rho(x) = \sup_{\lambda \in \sigma(x)} |\lambda| \leq \liminf_{n \to \infty} \|x^n\|^{1/n}.$$

Let $f \in \mathcal{A}'$ and $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$. Then

$$\begin{aligned} f(R(\lambda)) &= f\left((\lambda \mathbb{I} - x)^{-1}\right) = f\left(\lambda^{-1}(\mathbb{I} - \lambda^{-1} x)^{-1}\right) \\ &= f\left(\lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} x^n\right) \\ &= \lambda^{-1} \sum_{n=0}^{\infty} f(\lambda^{-n} x^n). \end{aligned}$$

This formula is true also when $|\lambda| > \rho(x)$, because $f \circ R$ is holomorphic in $\mathbb{C} \setminus \sigma(x) \supset \mathbb{C} \setminus \overline{\mathbb{D}(0, \rho(x))}$. Hence if we define $T_{\lambda,x,n} \in \mathcal{A}'' = \mathcal{L}(\mathcal{A}', \mathbb{C})$ by $T_{\lambda,x,n}(f) := f(\lambda^{-n} x^n)$, we obtain

$$\sup_{n \in \mathbb{N}} |T_{\lambda,x,n}(f)| = \sup_{n \in \mathbb{N}} |f(\lambda^{-n} x^n)| < \infty \qquad (\text{when } |\lambda| > \rho(x))$$

for every $f \in \mathcal{A}'$; the Banach–Steinhaus Theorem applied on the family $\{T_{\lambda,x,n}\}_{n \in \mathbb{N}}$ shows that

$$M_{\lambda,x} := \sup_{n \in \mathbb{N}} \|T_{\lambda,x,n}\| < \infty,$$

so that we have

$$\begin{aligned} \|\lambda^{-n} x^n\| &\overset{\text{Hahn–Banach}}{=} \sup_{f \in \mathcal{A}' : \|f\| \leq 1} |f(\lambda^{-n} x^n)| \\ &= \sup_{f \in \mathcal{A}' : \|f\| \leq 1} |T_{\lambda,x,n}(f)| \\ &= \|T_{\lambda,x,n}\| \\ &\leq M_{\lambda,x}. \end{aligned}$$

Hence

$$\|x^n\|^{1/n} \leq M_{\lambda,x}^{1/n} |\lambda| \to_{n \to \infty} |\lambda|,$$

when $|\lambda| > \rho(x)$. Thus

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \le \rho(x);$$

collecting the results, the Spectral Radius Formula is verified $\square$

Remark 1. The Spectral Radius Formula contains startling information: the spectral radius $\rho(x)$ is purely an algebraic property (though related to a topological algebra), but the quantity $\lim \|x^n\|^{1/n}$ relies on both algebraic and metric properties! Yet the results are equal!

Remark 2. $\rho(x)^{-1}$ is the radius of convergence of the $\mathcal{A}$-valued power series $\lambda \mapsto \sum_{n=0}^{\infty} \lambda^n x^n$.

Remark 3. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ its Banach subalgebra. If $x \in \mathcal{B}$ then

$$\sigma_{\mathcal{A}}(x) \subset \sigma_{\mathcal{B}}(x)$$

and the inclusion can be proper, but the spectral radii for both Banach algebras are the same, since

$$\rho_{\mathcal{A}}(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \rho_{\mathcal{B}}(x).$$

Exercise. Let $\mathcal{A}$ be a Banach algebra, $x, y \in \mathcal{A}$. Show that $\rho(xy) = \rho(yx)$. Show that if $x \in \mathcal{A}$ is *nilpotent* (i.e. $x^k = 0$ for some $k \in \mathbb{N}$) then $\sigma(x) = \{0\}$. Give examples of nilpotent linear operators.

Exercise. Let $\mathcal{A}$ be a Banach algebra and $x, y \in \mathcal{A}$ such that $xy = yx$. Prove that $\rho(xy) \le \rho(x)\rho(y)$.