# Appendix on functional analysis

Let $X, Y$ be normed spaces with norms $x \mapsto \|x\|_X$ and $y \mapsto \|y\|_Y$, respectively. The set of bounded linear mappings $X \to Y$ is denoted by $\mathcal{L}(X, Y)$; the operator norm $(A \mapsto \|A\|) : \mathcal{L}(X, Y) \to \mathbb{R}$ is defined by

$$\|A\| := \sup_{x \in X : \|x\|_X \leq 1} \|Ax\|_Y.$$

Let us denote the dual of a normed space $X$ by $X' := \mathcal{L}(X, \mathbb{C})$.

**Hahn–Banach Theorem.** *Let $X$ be a normed vector space, $M \subset X$ be a vector subspace, and $f : M \to \mathbb{C}$ a bounded linear functional. Then there is a bounded linear functional $F : X \to \mathbb{C}$ such that $\|f\| = \|F\|$ and $f(x) = F(x)$ for every $x \in M$* □

**Corollary.** *Let $X$ is a normed space. Then*

$$\|x\| = \max_{F \in X' : \|F\| \leq 1} |F(x)|$$

*for every $x \in X$* □

**Banach–Steinhaus Theorem (Uniform Boundedness Principle).** *Let $X, Y$ be Banach spaces and $\{T_j\}_{j \in J} \subset \mathcal{L}(X, Y)$. If*

$$\sup_{j \in J} \|T_j x\| < \infty$$

*for every $x \in X$ then $\sup_{j \in J} \|T_j\| < \infty$* □