4 Topological algebras

Topological algebra. A topological space and an algebra $\mathcal{A}$ is called a *topological algebra* if

1. $\{0\} \subset \mathcal{A}$ is a closed subset, and
2. the algebraic operations are continuous, i.e. the mappings

$$\begin{aligned} ((\lambda, x) \mapsto \lambda x) : &\quad \mathbb{C} \times \mathcal{A} \to \mathcal{A}, \\ ((x, y) \mapsto x + y) : &\quad \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \\ ((x, y) \mapsto xy) : &\quad \mathcal{A} \times \mathcal{A} \to \mathcal{A} \end{aligned}$$

   are continuous.

Remark 1. Similarly, a *topological vector space* is a topological space and a vector space, in which $\{0\}$ is a closed subset and the vector space operations $(\lambda, x) \mapsto \lambda x$ and $(x, y) \mapsto x + y$ are continuous. And the reader might now guess how to define for instance a *topological group*...

Remark 2. Some books omit the assumption that $\{0\}$ should be a closed set; then e.g. any algebra $\mathcal{A}$ with a topology $\tau = \{\emptyset, \mathcal{A}\}$ would become a topological algebra. However, such generalizations are seldom useful. And it will turn out soon, that actually our topological algebras are indeed Hausdorff spaces! $\{0\}$ being a closed set puts emphasis on closed ideals and continuous homomorphisms, as we shall see later in this section.

Examples of topological algebras.

1. The commutative algebra $\mathbb{C}$ endowed with its usual topology (given by the absolute value norm $x \mapsto |x|$) is a topological algebra.
2. If $(X, x \mapsto \|x\|)$ is a normed space, $X \neq \{0\}$, then $\mathcal{L}(X)$ is a topological algebra with the norm

$$A \mapsto \|A\| := \sup_{x \in X: \|x\| \leq 1} \|Ax\|.$$

   Notice that $\mathcal{L}(\mathbb{C}) \cong \mathbb{C}$, and $\mathcal{L}(X)$ is non-commutative if $\dim(X) \geq 2$.

3. Let X be a set. Then

$$\mathcal{F}_b(X) := \{f \in \mathcal{F}(X) \mid f \text{ is bounded}\}$$

is a commutative topological algebra with the supremum norm

$$f \mapsto \|f\| := \sup_{x \in X} |f(x)|.$$

Similarly, if X is a topological space then the algebra

$$C_b(X) := \{f \in C(X) \mid f \text{ is bounded}\}$$

of bounded continuous functions on X is a commutative topological algebra when endowed with the supremum norm.

4. If (X, d) is a metric space then the algebra

$$\mathrm{Lip}(X) := \{f : X \to \mathbb{C} \mid f \text{ is Lipschitz continuous and bounded}\}$$

is a commutative topological algebra with the norm

$$f \mapsto \|f\| := \max \left\{ \sup_{x \in X} |f(x)|,\ \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \right\}.$$

5. $\mathcal{E}(\mathbb{R}) := C^\infty(\mathbb{R})$ is a commutative topological algebra with the metric

$$(f, g) \mapsto \sum_{m=1}^{\infty} 2^{-m} \frac{p_m(f-g)}{1 + p_m(f-g)}, \quad \text{where } p_m(f) := \max_{|x| \leq m, k \leq m} |f^{(k)}(x)|.$$

This algebra is not normable.

6. The topological dual $\mathcal{E}'(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$, the so called space of compactly supported distributions. There the multiplication is the convolution, which is defined for nice enough f, g by

$$(f, g) \mapsto f * g, \quad (f * g)(x) := \int_{-\infty}^{\infty} f(x - y)\ g(y)\ \mathrm{d}y.$$

The unit element of $\mathcal{E}(\mathbb{R})$ is the Dirac delta distribution δ_0 at the origin $0 \in \mathbb{R}$. This is a commutative topological algebra with the weak*-topology, but it is not metrizable.

7. Convolution algebras of compactly supported distributions on Lie groups are non-metrizable topological algebras; such an algebra is commutative if and only if the group is commutative.

Remark. Let $\mathcal{A}$ be a topological algebra, $U \subset \mathcal{A}$ open, and $S \subset \mathcal{A}$. Due to the continuity of $((\lambda, x) \mapsto \lambda x) : \mathbb{C} \times \mathcal{A} \to \mathcal{A}$ the set $\lambda U = \{\lambda u \mid u \in U\}$ is open if $\lambda \neq 0$. Due to the continuity of $((x, y) \mapsto x + y) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ the set $U + S = \{u + s \mid u \in U,\ s \in S\}$ is open.

Exercise. Topological algebras are Hausdorff spaces.

Remark. Notice that in the previous exercise you actually need only the continuities of the mappings $(x, y) \mapsto x + y$ and $x \mapsto -x$, and the fact that $\{0\}$ is a closed set. Indeed, the commutativity of the addition operation is not needed, so that you can actually prove a proposition "Topological groups are Hausdorff spaces"!

Exercise*. Let $\mathcal{A}$ be an algebra and a normed space. Prove that it is a topological algebra if and only if there exists a constant $C < \infty$ such that

$$\|xy\| \leq C \ \|x\| \ \|y\|$$

for every $x, y \in \mathcal{A}$.

Closed ideals

In topological algebras, the good ideals are the closed ones.

Examples. Let $\mathcal{A}$ be a topological algebra; then $\{0\} \subset \mathcal{A}$ is a closed ideal. Let $\mathcal{B}$ be another topological algebra, and $\phi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ be **continuous**. Then it is easy to see that $\mathrm{Ker}(\phi) = \phi^{-1}(\{0\}) \subset \mathcal{A}$ is a **closed** ideal; this is actually a canonical example of closed ideals.

Proposition. *Let $\mathcal{A}$ be a topological algebra and $\mathcal{J}$ its ideal. Then either $\overline{\mathcal{J}} = \mathcal{A}$ or $\overline{\mathcal{J}} \subset \mathcal{A}$ is a closed ideal.*

Proof. Let $\lambda \in \mathbb{C}$, $x, y \in \overline{\mathcal{J}}$, and $z \in \mathcal{A}$. Take $V \in \mathcal{V}(\lambda x)$. Then there exists $U \in \mathcal{V}(x)$ such that $\lambda U \subset V$ (due to the continuity of the multiplication by a scalar). Since $x \in \overline{\mathcal{J}}$, we may pick $x_0 \in \mathcal{J} \cap U$. Now

$$\lambda x_0 \in \mathcal{J} \cap (\lambda U) \subset \mathcal{J} \cap V,$$

which proves that $\lambda x \in \overline{\mathcal{J}}$. Next take $W \in \mathcal{V}(x+y)$. Then for some $U \in \mathcal{V}(x)$ and $V \in \mathcal{V}(y)$ we have $U + V \subset W$ (due to the continuity of the mapping $(x, y) \mapsto x + y$). Since $x, y \in \overline{\mathcal{J}}$, we may pick $x_0 \in \mathcal{J} \cap U$ and $y_0 \in \mathcal{J} \cap V$. Now

$$x + y \in \mathcal{J} \cap (U + V) \subset \mathcal{J} \cap W,$$

which proves that $x + y \in \overline{\mathcal{J}}$. Finally, we should show that $xz, zx \in \overline{\mathcal{J}}$, but this proof is so similar to the previous steps that it is left for the reader as an easy task $\square$

Topology for quotient algebra. Let $\mathcal{J}$ be an ideal of a topological algebra $\mathcal{A}$. Let τ be the topology of $\mathcal{A}$. For $x \in \mathcal{A}$, define $[x] = x + \mathcal{J}$, and let $[S] = \{[x] \mid x \in S\}$. Then it is easy to check that $\{[U] \mid U \in \tau\}$ is a topology of the quotient algebra $\mathcal{A}/\mathcal{J}$; it is called the *quotient topology.*

Remark. Let $\mathcal{A}$ be a topological algebra and $\mathcal{J}$ be its ideal. The quotient map $(x \mapsto [x]) \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}/\mathcal{J})$ is continuous: namely, if $x \in \mathcal{A}$ and $[V] \in \mathcal{V}_{\mathcal{A}/\mathcal{J}}([x])$ for some $V \in \tau$ then $U := V + \mathcal{J} \in \mathcal{V}(x)$ and $[U] = [V]$.

Lemma. *Let $\mathcal{J}$ be an ideal of a topological algebra $\mathcal{A}$. Then the algebra operations on the quotient algebra $\mathcal{A}/\mathcal{J}$ are continuous.*

Proof. Let us check the continuity of the multiplication in the quotient algebra: Suppose $[x][y] = [xy] \in [W]$, where $W \subset \mathcal{A}$ is an open set (recall that every open set in the quotient algebra is of the form $[W]$). Then

$$xy \in W + \mathcal{J}.$$

Since $\mathcal{A}$ is a topological algebra, there are open sets $U \in \mathcal{V}_{\mathcal{A}}(x)$ and $V \in \mathcal{V}_{\mathcal{A}}(y)$ satisfying

$$UV \subset W + \mathcal{J}.$$

Now $[U] \in \mathcal{V}_{\mathcal{A}/\mathcal{J}}([x])$ and $[V] \in \mathcal{V}_{\mathcal{A}/\mathcal{J}}([y])$. Furthermore, $[U][V] \subset [W]$ because

$$(U + \mathcal{J})(V + \mathcal{J}) \subset UV + \mathcal{J} \subset W + \mathcal{J};$$

we have proven the continuity of the multiplication $([x], [y]) \mapsto [x][y]$. As an easy exercise, we leave it for the reader to verify the continuities of the mappings $(\lambda, [x]) \mapsto \lambda[x]$ and $([x], [y]) \mapsto [x] + [y]$ $\square$

Exercise. Complete the previous proof by showing the continuities of the mappings $(\lambda, [x]) \mapsto \lambda[x]$ and $([x], [y]) \mapsto [x] + [y]$.

With the previous Lemma, we conclude:

Proposition. *Let $\mathcal{J}$ be an ideal of a topological algebra $\mathcal{A}$. Then $\mathcal{A}/\mathcal{J}$ is a topological algebra if and only if $\mathcal{J}$ is closed.*

Proof. If the quotient algebra is a topological algebra then $\{[0]\} = \{\mathcal{J}\}$ is a closed subset of $\mathcal{A}/\mathcal{J}$; since the quotient homomorphism is a continuous mapping, $\mathcal{J} = \mathrm{Ker}(x \mapsto [x]) \subset \mathcal{A}$ must be a closed set.

Conversely, suppose $\mathcal{J}$ is a closed ideal of a topological algebra $\mathcal{A}$. Then we deduce that

$$(\mathcal{A}/\mathcal{J}) \setminus \{[0]\} = [\mathcal{A} \setminus \mathcal{J}]$$

is an open subset of the quotient algebra, so that $\{[0]\} \subset \mathcal{A}/\mathcal{J}$ is closed $\square$

Remark. Let X be a topological vector space and M be its subspace. The reader should be able to define the *quotient topology* for the quotient vector space $X/M = \{[x] := x + M \mid x \in X\}$. Now X/M is a topological vector space if and only if M is a closed subspace.

Let $M \subset X$ be a closed subspace. If d is a metric on X then there is a natural metric for X/M:

$$([x],[y]) \mapsto d([x],[y]) := \inf_{z \in M} d(x-y, z),$$

and if X is a complete metric space then X/M is also complete. Moreover, if $x \mapsto \|x\|$ is a norm on X then there is a natural norm for X/M:

$$[x] \mapsto \|[x]\| := \inf_{z \in M} \|x - z\|.$$