12 Algebras of Lipschitz functions

This section is devoted to metric properties, not merely metrizability. We shall study how to recover the metric space structure from a normed algebra of Lipschitz functions in the spirit of the Gelfand theory of commutative Banach algebras. In the sequel, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Lipschitz mappings. Let $(X, d_X), (Y, d_Y)$ be metric spaces; often we drop the subscripts from metrics, i.e. write d for both d_X and d_Y without confusion. A mapping $f : X \to Y$ is called *Lipschitz* if

$$\exists C < \infty \ \forall x, y \in X : \ d_Y(f(x), f(y)) \leq C \ d_X(x, y);$$

then the *Lipschitz constant of f* is

$$\begin{aligned} L(f) &:= \inf\{C \in \mathbb{R} \mid \forall x, y \in X : \ d(f(x), f(y)) \leq C \ d(x, y)\} \\ &= \sup_{x,y \in X: \ x \neq y} \frac{d(f(x), f(y))}{d(x, y)}. \end{aligned}$$

A mapping $f : X \to Y$ is called *bi-Lipschitz* (or a *quasi-isometry*) if it is bijective and f, f^{-1} are both Lipschitz.

Examples.

1. Lipschitz mappings are uniformly continuous, but not the vice versa: for instance, $(t \mapsto \sqrt{t}) : [0, 1] \to \mathbb{R}$ is uniformly continuous, but not Lipschitz.

2. The distance from $x \in X$ to a non-empty set $A \subset X$ is defined by

   $$d(x, A) = d(A, x) := \inf_{a \in A} d(x, a),$$

   Then $d_A = (x \mapsto d(x, A)) : X \to \mathbb{R}$ is a Lipschitz mapping, $L(d_A) \leq 1$; notice that $d_A(x) = 0$ if and only if $x \in \overline{A}$. Thus there are plenty of Lipschitz functions on a metric space.

Exercise*. Let $A, B \subset X$ be non-empty sets. Assume that the distance between A, B is positive, i.e. $d(A, B) > 0$, where

$$d(A, B) := \inf_{a \in A, \ b \in B} d(a, b).$$

Show that there exists a Lipschitz function $f : X \to \mathbb{R}$ such that

$$0 \leq f \leq 1, \quad f(A) = \{0\}, \quad f(B) = \{1\}.$$

This is the Lipschitz analogy of Urysohn's Lemma.

Tietze's Extension Theorem (Lipschitz analogy). *Let X be a metric space, $A \subset X$ non-empty, and $f : A \to \mathbb{K}$ bounded. Then there exists $F : X \to \mathbb{K}$ such that*

$$F|_A = f, \quad \|F\|_{C(X)} = \|f\|_{C(A)}, \quad \begin{cases} L(F) = L(f), & \text{when } \mathbb{K} = \mathbb{R}, \\ L(F) \leq \sqrt{2}\ L(f), & \text{when } \mathbb{K} = \mathbb{C}. \end{cases}$$

Proof. Here $\|f\|_{C(A)} := \sup_{x \in A} |f(x)|$. When $L(f) = \infty$, define $F : X \to \mathbb{K}$ by $F|_A = f$, $F(X \setminus A) = \{0\}$. For the rest of the proof, suppose $L(f) < \infty$.

Let us start with the case $\mathbb{K} = \mathbb{R}$. Define $G : X \to \mathbb{R}$ by

$$G(x) = \inf_{a \in A} (f(a) + L(f)\ d(x,a)),$$

so that $G|_A = f$, as the reader may verify. Define $F : X \to \mathbb{R}$ by

$$F(x) = \begin{cases} G(x), & \text{when } |G(x)| \leq \|f\|_{C(A)}, \\ \|f\|_{C(A)}\ \frac{G(x)}{|G(x)|}, & \text{when } |G(x)| > \|f\|_{C(A)}. \end{cases}$$

Clearly $F|_A = f$, $\|F\|_{C(X)} = \|f\|_{C(A)}$, and $L(f) \leq L(F) \leq L(G)$; let us then show that $L(G) = L(f)$. Suppose $x, y \in X$. Take $\varepsilon > 0$. Choose $a_\varepsilon \in A$ such that $G(y) \geq f(a_\varepsilon) + L(f)\ d(y, a_\varepsilon) - \varepsilon$. Then

$$\begin{aligned} G(x) - G(y) &= \inf_{a \in A} (f(a) + L(f)\ d(x,a)) - G(y) \\ &\leq (f(a_\varepsilon) + L(f)\ d(x, a_\varepsilon)) - (f(a_\varepsilon) + L(f)\ d(y, a_\varepsilon) - \varepsilon) \\ &= L(f)\ (d(x, a_\varepsilon) - d(y, a_\varepsilon)) + \varepsilon \\ &\leq L(f)\ d(x,y) + \varepsilon, \end{aligned}$$

which yields $G(x) - G(y) \leq L(f)\ d(x,y)$. Symmetrically, $G(y) - G(x) \leq L(f)\ d(x,y)$, so that $|G(x) - G(y)| \leq L(f)\ d(x,y)$. Hence we have proven that $L(G) \leq L(f)$, which completes the proof of the case $\mathbb{K} = \mathbb{R}$.

Let us consider the case $\mathbb{K} = \mathbb{C}$. Let $f_1 = \Re(f)$, $f_2 = \Im(f)$. Then using the $\mathbb{R}$-result we can extend $f_1, f_2 : A \to \mathbb{R}$ to functions $F_1, F_2 : X \to \mathbb{R}$ satisfying

$$F_j|_A = f_j, \quad L(F_j) = L(f_j) \leq L(f), \quad \|F_j\|_{C(X)} = \|f_j\|_{C(A)}.$$

Let us define $G : X \to \mathbb{C}$ by $G = F_1 + iF_2$, and define $F : X \to \mathbb{C}$ by

$$F(x) = \begin{cases} G(x), & \text{when } |G(x)| \leq \|f\|_{C(A)}, \\ \|f\|_{C(A)}\ \frac{G(x)}{|G(x)|}, & \text{when } |G(x)| > \|f\|_{C(A)}. \end{cases}$$

Then $\|F\|_{C(X)} = \|f\|_{C(A)}$. Moreover, we obtain $L(G) \leq \sqrt{2}\ L(f)$, because $|z| \leq \sqrt{2}\ \max\{|\Re(z)|, |\Im(z)|\}$ for every $z \in \mathbb{C}$; hence

$$L(f) \leq L(F) \leq L(G) \leq \sqrt{2}\ L(f),$$

completing the proof $\square$

Lipschitz spaces. Let X be a metric space. Let

$$\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathbb{K}) := \{f : X \to \mathbb{K} :\ \|f\|_{\mathrm{Lip}} = \max(\|f\|_{C(X)}, L(f)) < \infty\}.$$

A *pointed metric space* is a metric space X with a distinguished element, the *base point* $e_X = e \in X$; let

$$\mathrm{Lip}_0(X) = \mathrm{Lip}_0(X, \mathbb{K}) := \{f : X \to \mathbb{K} \mid f(e) = 0,\ L(f) < \infty\}.$$

Notice that if the diameter $\mathrm{diam}(X) = \sup_{x,y \in X} d(x, y)$ of the space is finite then $\mathrm{Lip}_0(X)$ is contained in $\mathrm{Lip}(X)$,

Exercise*. Show that $\mathrm{Lip}(X)$ is a Banach space with the norm $f \mapsto \|f\|_{\mathrm{Lip}}$. Show that $\mathrm{Lip}_0(X)$ is a Banach space with the norm $f \mapsto L(f)$. Show that these spaces are topological algebras if $\mathrm{diam}(X) < \infty$.

Arens–Eells space. Let X be a metric space, $x, y \in X$. The *xy-atom* is the function $m_{xy} : X \to \mathbb{K}$ defined by

$$m_{xy}(x) = 1, \quad m_{xy}(y) = -1, \quad m_{xy}(z) = 0 \text{ otherwise}.$$

A *molecule on* X is a linear combination $m = \sum_{j=1}^{n} a_j\ m_{x_j y_j}$ of such atoms; then $\{x \in X \mid m(x) \neq 0\}$ is a finite set and $\sum_{x \in X} m(x) = 0$. Let M denote the $\mathbb{K}$-vector space of the molecules on X. Notice that a molecule may have several representations as a linear combination of atoms. Let us define a mapping $m \mapsto \|m\|_{AE} : M \to \mathbb{R}$ by

$$\|m\|_{AE} := \inf \left\{ \sum_{j=1}^{n} |a_j|\ d(x_j, y_j) :\ n \in \mathbb{Z}^+,\ m = \sum_{j=1}^{n} a_j\ m_{x_j y_j} \right\};$$

obviously this is a seminorm on the space of the molecules, but we shall prove that it is actually a norm; for the time being, we have to define the *Arens–Eells space* $AE(X)$ for X by completing the vector space M with respect to the Arens–Eells-**seminorm** $m \mapsto \|m\|_{AE}$ modulo the subspace $\{v :\ \|v\|_{AE} = 0\}$.

Theorem. *The Banach space dual of $AE(X)$ is isometrically isomorphic to* $\mathrm{Lip}_0(X)$.

Proof. Let us define two linear mappings $T_1 : AE(X)' \to \mathrm{Lip}_0(X)$ and $T_2 : \mathrm{Lip}_0(X) \to AE(X)'$ by

$$(T_1\phi)(x) := \phi(m_{xe}), \quad (T_2 f)(m) := \sum_{y\in X} f(y)\ m(y),$$

where $e \in X$ is the base point, and $m \in M$ is a molecule (so that $T_2 f$ is uniquely extended to a linear functional on $AE(X)$). These definitions are sound indeed: Firstly,

$$(T_1\phi)(e) = \phi(m_{ee}) = \phi(0) = 0,$$

$$\begin{aligned} |(T_1\phi)(x) - (T_1\phi)(y)| &= |\phi(m_{xe} - m_{ye})| = |\phi(m_{xy})| \leq \|\phi\|\ \|m_{xy}\|_{AE} \\ &\leq \|\phi\|\ d(x,y), \end{aligned}$$

so that $T_1\phi \in \mathrm{Lip}_0(X)$ and $L(T_1\phi) \leq \|\phi\|$; we have even proven that $T_1 \in \mathcal{L}(AE(X)', \mathrm{Lip}_0(X))$ with norm $\|T_1\| \leq 1$. Secondly, let $\varepsilon > 0$ and $m \in M$. We may choose $(a_j)_{j=1}^n \subset \mathbb{K}$ and $((x_j,y_j))_{j=1}^n \subset X \times X$ such that

$$m = \sum_{j=1}^n a_j\ m_{x_j y_j}, \quad \sum_{j=1}^n |a_j|\ d(x_j,y_j) \leq \|m\|_{AE} + \varepsilon.$$

Then

$$\begin{aligned} |(T_2 f)(m)| &= \left|(T_2 f)\ \sum_{j=1}^n a_j\ m_{x_j y_j}\right| = \left|\sum_{j=1}^n a_j\ (f(x_j) - f(y_j))\right| \\ &\leq \sum_{j=1}^n |a_j|\ |f(x_j) - f(y_j)| \\ &\leq L(f)\ \sum_{j=1}^n |a_j|\ d(x_j,y_j) \\ &\leq L(f)\ (\|m\|_{AE} + \varepsilon), \end{aligned}$$

meaning that $T_2 \in \mathcal{L}(\mathrm{Lip}_0(X), AE(X)')$ with norm $\|T_2\| \leq 1$. Next we notice that $T_2 = T_1^{-1}$:

$$(T_1(T_2 f))(x) = (T_2 f)(m_{xe}) = \sum_{y\in X} f(y)\ m_{xe}(y) = f(x) - f(e) = f(x),$$

$$
\begin{aligned}
(T_2(T_1\phi))(m) &= \sum_{y\in X}(T_1\phi)(y)\ m(y) = \sum_{y\in X}\phi(m_{ye})\ m(y) \\
&= \phi\left(\sum_{y\in X} m(y)\ m_{ye}\right) = \phi(m).
\end{aligned}
$$

Finally, for $f \in \mathrm{Lip}_0(X)$ we have

$$
L(f) = L(T_1T_2f) \leq \|T_1\|\ \|T_2f\| \leq \|T_2f\| \leq \|T_2\|L(f) \leq L(f),
$$

so that $T_2, T_1 = T_2^{-1}$ are isometries $\square$

Remark. Let us denote

$$
((f,m) \mapsto \langle f,m\rangle) : \mathrm{Lip}_0(X) \times AE(X) \to \mathbb{K}
$$

where

$$
\langle f,m\rangle = \sum_{x\in X} f(x)\ m(x)
$$

if $m \in M$. From now on, the weak*-topology of $\mathrm{Lip}_0(X)$ refers to the $AE(X)$-induced topology, with the interpretation

$$
AE(X) \subset AE(X)'' \cong \mathrm{Lip}_0(X)'.
$$

Next we show how X is canonically embedded in the Arens–Eells space:

Corollary. *The Arens–Eells seminorm $m \mapsto \|m\|_{AE}$ is a norm, and the mapping $(x \mapsto m_{xe}) : X \to AE(X)$ is an isometry.*

Proof. Take $m \in M$, $m \neq 0$. Choose $x_0 \in X$ such that $m(x_0) \neq 0$. Due to the theorem above,

$$
\|m\|_{AE} \overset{\text{Hahn–Banach}}{=} \sup_{f\in AE(X)':\|f\|\leq 1} |\langle f,m\rangle| = \sup_{f\in \mathrm{Lip}_0(X):L(f)\leq 1}\left|\sum_{x\in X} f(x)\ m(x)\right|.
$$

Let $A := \{e\}\cup\{x \in X \mid m(x) \neq 0\}$. Let $r := d(x_0, A\setminus\{x_0\})$. By the Lipschitz analogy of Tietze's Extension Theorem, there exists $f_0 \in \mathrm{Lip}_0(X,\mathbb{R})$ such that $f_0(x_0) = r > 0$, $f_0(A \setminus \{x_0\}) = \{0\}$, and $L(f_0) = 1$. Thereby

$$
\|m\|_{AE} \geq |\langle f_0,m\rangle| = \left|\sum_{x\in X} f_0(x)\ m(x)\right| = |f_0(x_0)\ m(x_0)| > 0,
$$

i.e. $m \mapsto \|m\|_{AE}$ is actually a norm.

Let $x, y \in X$. Clearly $\|m_{xy}\|_{AE} \leq d(x,y)$. Define $\widetilde{d_y}(z) := d(z,y) - d(e,y)$, where $e \in X$ is the base point. Now $\widetilde{d_y} \in \mathrm{Lip}_0(X)$ and $L(\widetilde{d_y}) = 1$, so that

$$\begin{aligned} \|m_{xy}\|_{AE} \quad &\geq \quad |\langle \widetilde{d_y}, m_{xy} \rangle| \quad = \quad \left| \sum_{z \in X} \widetilde{d_y}(z)\ m_{xy}(z) \right| \quad = \quad |\widetilde{d_y}(x) - \widetilde{d_y}(y)| \\ &= \quad d(x,y). \end{aligned}$$

Hence $\|m_{xe} - m_{ye}\|_{AE} = \|m_{xy}\|_{AE} = d(x,y)$ $\square$

Nets and convergence. A partial order $(J, \leq)$ is called a *directed* if

$$\forall i, j \in J\ \exists k \in J:\ i \leq k,\ j \leq k.$$

A *net* in a topological space (X, τ) is a family $(x_j)_{j \in J} \subset X$, where $J = (J, \leq)$ is directed. A net $(x_j)_{j \in J} \subset X$ *converges to a point* $x \in X$, denoted by

$$x_j \to x \quad \text{or} \quad x_j \to_{j \in J} x \quad \text{or} \quad x = \lim x_j = \lim_{j \in J} x_j,$$

if for every $U \in \mathcal{V}_\tau(x)$ there exists $j_U \in J$ such that $x_j \in U$ whenever $j_U \leq j$.

An example of a net is a *sequence* $(x_n)_{n \in \mathbb{N}} \subset X$, where $\mathbb{N}$ has the usual partial order; sequences characterize topology in spaces of countable local bases, for instance metric spaces. But there are more complicated topologies, where sequences are not enough; for example, weak*-topology for infinite-dimensional spaces.

Exercise*. Nets can be used to characterize the topology: Let (X, τ) be a topological space and $A \subset X$. Show that $x \in \overline{A} \subset X$ if and only if there exists a net $(x_j)_{j \in J} \subset A$ such that $x_j \to x$. Let $f : X \to Y$; show that $f \in C(X, Y)$ if and only if $x_j \to x \in X \ \Rightarrow f(x_j) \to f(x) \in Y$.
(Hint: define a partial order relation on $\mathcal{V}_\tau(x)$ by $U \leq V \ \Leftrightarrow\ V \subset U$.)

Lemma. *Let E be a Banach space. The weak*-converging nets in E' are bounded.*

Proof. Let $f_j \to f$ in the weak*-topology of E', i.e. $\langle f_j, \phi \rangle \to \langle f, \phi \rangle \in \mathbb{K}$ for every $\phi \in E$. Define $T_j : E \to \mathbb{K}$ by $\phi \mapsto \langle f_j, \phi \rangle$. Since $T_j \phi \to \langle f, \phi \rangle \in \mathbb{K}$, we have $\sup_{j \in J} |T_j \phi| < \infty$ for every $\phi \in E$, so that $C := \sup_{j \in J} \|T_j\| < \infty$ according to the Banach–Steinhaus Theorem. Thereby

$$\|f_j\| \overset{\text{Hahn–Banach}}{=} \sup_{\phi \in E : \|\phi\| \leq 1} |\langle f_j, \phi \rangle| = \sup_{\phi \in E : \|\phi\| \leq 1} |T_j \phi| \overset{\text{Hahn–Banach}}{=} \|T_j\| \leq C,$$

so that the net $(f_j)_{j \in J} \subset E'$ is bounded $\square$

Proposition. *On bounded subsets of* $\mathrm{Lip}_0(X)$ *the weak***-topology is the topology of pointwise convergence. Moreover, if* X *is compact, on bounded sets these topologies coincide with the topology of uniform convergence.*

Proof. Let $\mathcal{E} \subset \mathrm{Lip}_0(X)$ be a bounded set containing a net $(f_j)_{j\in J}$ such that $f_j \to f$ in the weak*-topology. Endow the norm-closure $\overline{\mathcal{E}}$ with the relative weak*-topology τ_1, and also with the topology τ_2 of pointwise convergence. If $x \in X$ then

$$f_j(x) = f_j(x) - f_j(e) = \langle f_j, m_{xe}\rangle \to \langle f, m_{xe}\rangle = f(x) - f(e) = f(x),$$

i.e. $f_j \to f$ pointwise. This means that the topology of pointwise convergence is weaker than the weak*-topology, $\tau_2 \subset \tau_1$. Now τ_1 is compact due to the Banach–Alaoglu Theorem, and of course τ_2 is Hausdorff; hence $\tau_1 = \tau_2$, the weak*-topology and the topology of the pointwise convergence coincide on bounded subsets.

Now suppose X is a compact metric space. Uniform convergence trivially implies pointwise convergence. Let $(f_j)_{j\in J} \subset \mathcal{E}$ be as above, $f_j \to f$ pointwise. Since $\mathcal{E}$ is bounded, there exists $C < \infty$ such that $L(g) \leq C$ for every $g \in \mathcal{E}$. It is easy to check that $L(f) \leq C$. Take $\varepsilon > 0$. Since X is compact, there exists $\{x_k\}_{k=1}^{n_\varepsilon} \subset X$ such that

$$\forall x \in X\ \exists k \in \{1, \cdots, n_\varepsilon\} :\ d(x, x_k) < \varepsilon.$$

Due to the pointwise convergence $f_j \to f$, there exists $j_\varepsilon \in J$ such that

$$|f_j(x_k) - f(x_k)| < \varepsilon$$

for every $k \in \{1, \cdots, n_\varepsilon\}$ whenever $j_\varepsilon \leq j$. Take $x \in X$. Take $k \in \{1, \cdots, n_\varepsilon\}$ such that $d(x, x_k) < \varepsilon$. Then

$$\begin{aligned} |f_j(x) - f(x)| &\leq |f_j(x) - f_j(x_k)| + |f_j(x_k) - f(x_k)| + |f(x_k) - f(x)| \\ &\leq L(f_j)\ d(x, x_k) + \varepsilon + L(f)\ d(x_k, x) \\ &\leq C\ \varepsilon + \varepsilon + C\ \varepsilon\ =\ (2C+1)\ \varepsilon. \end{aligned}$$

Thereby $\|f_j - f\|_{C(X)} \to 0$; pointwise convergence on bounded subsets implies uniform convergence, when X is compact $\square$

Algebra $\mathrm{Lip}_0(X)$**.** Let X be a metric space such that $\mathrm{diam}(X) < \infty$. In the sequel, we shall call $\mathrm{Lip}_0(X)$ an *algebra*, even though $\mathbb{I} \notin \mathrm{Lip}_0(X)$. An *algebra homomorphism* between such non-unital algebras is a linear and multiplicative mapping; then even the 0-mapping is a homomorphism!

Proposition. *Let X, Y be metric spaces with finite diameters, with the respective base points e_X, e_Y. Let $g : Y \to X$ be a Lipschitz mapping such that $g(e_Y) = e_X$. Then the mapping*

$$L_g : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y), \quad f \mapsto f \circ g,$$

is an algebra homomorphism, and $\|L_g\| = L(g)$.

Proof. If $f \in \mathrm{Lip}_0(X)$ and $x, y \in Y$ then

$$|f(g(x)) - f(g(y))| \leq L(f)\ d(g(x), g(y)) \leq L(f)\ L(g)\ d(x, y).$$

Hence $L(L_g f) = L(f \circ g) \leq L(f)\ L(g)$, implying $\|L_g\| \leq L(g)$. Take $y_0 \in Y$. Define $f_0 \in \mathrm{Lip}_0(X)$ by $f_0(x) := d(x, g(y_0)) - d(e_X, g(y_0))$, so that $L(f_0) = 1$. Take $y \in Y$, $y \neq y_0$. Then

$$\begin{aligned} \|L_g\| &\geq L(L_g(f_0)) \\ &\geq \frac{|(L_g f_0)(y) - (L_g f_0)(y_0)|}{d(y, y_0)} = \frac{d(g(y), g(y_0))}{d(y, y_0)}, \end{aligned}$$

so that $\|L_g\| \geq L(g)$; hence $\|L_g\| = L(g)$.

If $\lambda \in \mathbb{K}$ and $f, h \in \mathrm{Lip}_0(X)$ then

$$\begin{aligned} L_g(\lambda f) &= (\lambda f) \circ g = \lambda\ (f \circ g) = \lambda\ L_g f, \\ L_g(f + h) &= (f + h) \circ g = f \circ g + h \circ g = L_g f + L_g h, \\ L_g(fh) &= (fh) \circ g = (f \circ g)(h \circ g) = (L_g f)(L_g h), \end{aligned}$$

so that L_g is a homomorphism □

Order-completeness. Non-empty $\mathcal{B} \subset \mathrm{Lip}(X, \mathbb{R})$ is called *order-complete* if

$$\sup \mathcal{G}, \inf \mathcal{G} \in \mathcal{B}$$

for every bounded family $\mathcal{G} \subset \mathcal{B}$. Here supremums and infimums are pointwise, naturally.

Uniform separation. A family $\mathcal{F} \subset \mathrm{Lip}_0(X)$ *separates uniformly the points of X* if

$$\exists C < \infty\ \forall x, y \in X\ \exists g \in \mathcal{F} :\ L(g) \leq C,\ |g(x) - g(y)| = d(x, y).$$

In a striking resemblance with the "classical" Stone–Weierstrass Theorem, we have the following:

Theorem (Lipschitz Stone–Weierstrass). *Let X be a compact metric space. Let $\mathcal{A}$ be an involutive, weak*-closed subalgebra of* $\mathrm{Lip}_0(X)$ *separating the points of X uniformly. Then* $\mathcal{A} = \mathrm{Lip}_0(X)$.

Proof. As in the proof of the "classical" Stone–Weierstrass Theorem, involutivity justifies our concentration on the $\mathbb{R}$-scalar case, where the involution is trivial, $f^* = f$. Hence we assume that $\mathcal{A}$ is a weak*-closed $\mathbb{R}$-subalgebra of $\mathrm{Lip}_0(X, \mathbb{R})$ separating the points of X uniformly.

Let us show that $\mathcal{B} = \mathcal{A} + \mathbb{R}\mathbb{I}$ is closed under the pointwise convergence of bounded nets. Let $(g_j)_{j \in J} \subset \mathcal{B}$ be a bounded net converging to pointwise to $g \in \mathrm{Lip}(X)$; here $g_j = f_j + \lambda_j \mathbb{I}$ with $f_j \in \mathcal{A}$ and $\lambda_j \in \mathbb{R}$. Especially

$$\lambda_j = f_j(e) + \lambda_j = g_j(e) \to g(e) \in \mathbb{R}.$$

Thus

$$f_j(x) = g_j(x) - \lambda_j \to g(x) - g(e) \in \mathbb{R},$$

i.e. $f_j \to g - g(e)\mathbb{I}$ pointwise. But $(f_j)_{j \in J} \subset \mathcal{A}$ is a bounded net, so that $f_j \to g - g(e)\mathbb{I}$ in the weak*-topology; since $\mathcal{A}$ is weak*-closed, $g - g(e)\mathbb{I} \in \mathcal{A}$. Thereby

$$g = (g - g(e)\mathbb{I}) + g(e)\mathbb{I} \in \mathcal{A} + \mathbb{R}\mathbb{I} = \mathcal{B};$$

$\mathcal{B}$ is closed under the pointwise convergence of bounded nets.

Let us show that $\mathcal{B}$ is order-complete. First, let $g \in \mathcal{B}$. Take $\varepsilon > 0$. Let

$$g_\varepsilon(x) := \sqrt{g(x)^2 + \varepsilon^2}.$$

By the Weierstrass Approximation Theorem, there exists a sequence $(P_{\varepsilon n})_{n=1}^{\infty}$ of real-valued polynomials such that $P_{\varepsilon n}(0) = \varepsilon$ and

$$P'_{\varepsilon n}(t) \to_{n \to \infty} \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{t^2 + \varepsilon^2} = \frac{t}{\sqrt{t^2 + \varepsilon^2}}$$

uniformly on $[-\|g\|_{C(X)}, \|g\|_{C(X)}]$; consequently, $(P_{\varepsilon n}(g))_{n=1}^{\infty} \subset \mathcal{B}$ is a bounded sequence, converging uniformly to g_ε; hence $P_{\varepsilon n}(g) \to g_\varepsilon$ also pointwise. Since $\mathcal{B}$ is closed under the pointwise convergence of bounded nets, we deduce $g_\varepsilon \in \mathcal{B}$; consequently, $(g_\varepsilon)_{0 < \varepsilon \leq 1}$ is a bounded net in $\mathcal{B}$, so that $g_0 := \lim_{\varepsilon \to 0^+} g_\varepsilon$ belongs to $\mathcal{B}$. But $g_0(x) = |g(x)|$, so that $g \in \mathcal{B}$ implies $|g| \in \mathcal{B}$. Therefore if $f, g \in \mathcal{B}$ then

$$\max(f, g) = \frac{f + g}{2} + \frac{|f + g|}{2}, \quad \min(f, g) = \frac{f + g}{2} - \frac{|f + g|}{2}$$

belong to $\mathcal{B}$. Let $\mathcal{G} \subset \mathcal{B}$ be a bounded non-empty family. Let $\mathcal{H} \subset \mathrm{Lip}(X, \mathbb{R})$ be the smallest family closed under taking maximums and minimums and containing $\mathcal{G}$. Now $\mathcal{H} \subset \mathcal{B}$, since $\mathcal{B}$ is closed under taking maximums and minimums. Moreover, $\mathcal{H}$ is bounded. Clearly

$$\sup \mathcal{G} = \sup \mathcal{H} \in \mathrm{Lip}(X, \mathbb{R}) \quad \text{and} \quad \inf \mathcal{G} = \inf \mathcal{H} \in \mathrm{Lip}(X, \mathbb{R}).$$

Let $g := \sup \mathcal{G} \in \mathrm{Lip}(X, \mathbb{R})$. Take $\varepsilon > 0$. For each $x \in X$ there exists $g_x \in \mathcal{G}$ such that $g(x) - \varepsilon < g_x(x)$. Due to the continuity of g_x, there exists $U_x \in \mathcal{V}(x)$ such that $g(y) - \varepsilon < g_x(y)$ for every $y \in U_x$. Then $\{U_x \mid x \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{U_{x_j} \mid 1 \le j \le n\}$. Let $h_\varepsilon := \max(g_{x_1}, \cdots, g_{x_n}) \in \mathcal{H}$. Then

$$g(x) - \varepsilon < h_\varepsilon(x) < g(x)$$

for every $x \in X$, so that $(h_\varepsilon)_{0<\varepsilon\le 1} \subset \mathcal{H} \subset \mathcal{B}$ is a bounded net, $h_\varepsilon \to_{\varepsilon \to 0^+} g$. Hence $\sup \mathcal{G} = g \in \mathcal{B}$, because $\mathcal{B}$ is closed under the pointwise convergence of bounded nets. Similarly one proves that $\inf \mathcal{G} \in \mathcal{B}$. Thus $\mathcal{B}$ is order-complete.

Take $f \in \mathrm{Lip}_0(X, \mathbb{R})$. We have to show that $f \in \mathcal{A}$. We may assume that $L(f) \le 1$. Due to the uniform separation, for every $x, y \in X$ there exists $g_{xy} \in \mathcal{A}$ such that $L(g_{xy}) \le C$ (C does not depend on $x, y \in X$) and $|g_{xy}(x) - g_{xy}(y)| = d(x, y)$. Since $|f(x) - f(y)| \le L(f)\ d(x, y) \le d(x, y)$ and since $\mathcal{A}$ is an algebra, there exists $h_{xy} \in \mathcal{A}$ satisfying $h_{xy}(x) - h_{xy}(y) = f(x) - f(y)$ and $L(h_{xy}) \le C$. Define $f_{xy} \in \mathcal{B}$ by

$$f_{xy} := h_{xy} - (h_{xy}(y) - f(y))\mathbb{I}.$$

Then $f_{xy}(x) = f(x)$ and $f_{xy}(y) = f(y)$, $L(f_{xy}) = L(h_{xy}) \le C$, and

$$\|f_{xy}\|_{C(X)} \le \|h_{xy}\|_{C(X)} + |h_{xy}(y)| + |f(y)| \le (2C + 1)\ r(X),$$

where $r(X) := \sup_{z \in X} d(z, e) < \infty$ is the "radius" of the space. The family $(f_{xy})_{x,y \in X} \subset \mathcal{B}$ is hence bounded; due to the order-completeness of $\mathcal{B}$,

$$f = \inf_{x \in X} \sup_{y \in X} f_{xy}$$

belongs to $\mathcal{B}$; but $f(e) = 0$, so that $f \in \mathcal{A}$ $\square$

Quotient metrics. Let X be a compact metric space and $\mathcal{A} \subset \mathrm{Lip}_0(X)$ be an involutive, weak*-closed subalgebra. Let $R_\mathcal{A}$ be the equivalence relation

$$(x, y) \in R_\mathcal{A} \overset{\text{definition}}{\Longleftrightarrow} \forall f \in \mathcal{A} : \ f(x) = f(y).$$

Let $[x] := \{y \in X \mid (x, y) \in R_\mathcal{A}\}$. Let us endow $X_\mathcal{A} := X/R_\mathcal{A} = \{[x] \mid x \in X\}$ with the metric

$$d_{X_\mathcal{A}}([x], [y]) := \sup_{f \in \mathcal{A} : L(f) \le 1} |f(x) - f(y)|.$$

Let $\pi = (x \mapsto [x]) : X \to X_\mathcal{A}$. Recall that this induces a homomorphism $L_\pi = (\tilde{f} \mapsto \tilde{f} \circ \pi) : \mathrm{Lip}_0(X_\mathcal{A}) \to \mathrm{Lip}_0(X)$.

Corollary. *Let X be a compact metric space, and let $\mathcal{A}$ be an involutive, weak*-closed subalgebra of* $\mathrm{Lip}_0(X)$. *Then* $L_\pi : \mathrm{Lip}_0(X_\mathcal{A}) \to \mathcal{A} \subset \mathrm{Lip}_0(X)$ *is a bounded algebra isomorphism* $\mathrm{Lip}_0(X_\mathcal{A}) \cong \mathcal{A}$ *with a bounded inverse.*

Exercise*. Prove the previous Corollary.

Exercise*. Show that weak*-closed ideals of $\mathrm{Lip}_0(X)$ are involutive, when X is compact. (Hint: Lipschitz–Stone–Weierstrass.)

Varieties and ideals. Let X be a metric space, $S \subset X$, and $\mathcal{J} \subset \mathrm{Lip}_0(X)$. Then

$$\mathcal{I}(S) := \{f \in \mathrm{Lip}_0(X) \mid \forall x \in S: \ f(x) = 0\}$$

is a weak*-closed ideal of $\mathrm{Lip}_0(X)$ (the *ideal of* S), and

$$V(\mathcal{J}) := \{x \in X \mid \forall f \in \mathcal{J}: \ f(x) = 0\}$$

is a closed subset of X (the *variety of* $\mathcal{J}$).

Theorem. *Let X be a compact metric space, $\mathcal{J}$ be a weak*-closed ideal of* $\mathrm{Lip}_0(X)$. *Then* $\mathcal{J} = \mathcal{I}(V(\mathcal{J}))$.

Exercise*. Prove the previous theorem. (Hint: show that $d(x, V(\mathcal{J})) = d_{X_\mathcal{J}}([x], V(\mathcal{J}))$ for every $x \in X$, use Lipschitz–Stone–Weierstrass.)

Corollary. *Let X be a compact metric space, and let $\omega : \mathrm{Lip}_0(X) \to \mathbb{K}$ be an algebra homomorphism. Then ω is weak*-continuous if and only if $\omega = \omega_x := (x \mapsto f(x))$ for some $x \in X$.*

Proof. If $\omega_x := (x \mapsto f(x)) : \mathrm{Lip}_0(X) \to \mathbb{K}$ then $\omega_x = m_{xe} \in AE(X)$ in the sense that $\langle f, \omega_x \rangle = f(x) = \langle f, m_{xe} \rangle$; hence evaluation homomorphisms are weak*-continuous.

Conversely, let $\omega : \mathrm{Lip}_0(X) \to \mathbb{K}$ be a weak*-continuous homomorphism. Then $\mathrm{Ker}(\omega)$ is an weak*-closed ideal of $\mathrm{Lip}_0(X)$, hence involutive. Thus by the previous Theorem $\mathrm{Ker}(\omega) = \mathcal{I}(V)$ for some $V \subset X$. Notice that $0 = \omega_e$; assume that $\omega \neq 0$. Since ω is a surjective linear mapping onto $\mathbb{K}$, $\mathrm{Ker}(\omega)$ must be of co-dimension 1 in $\mathrm{Lip}_0(X)$, and thereby $V = \{e, x\}$ for some $x \in X$. Hence $\omega = (f \mapsto \lambda f(x))$ for some $\lambda \in \mathbb{K}$, $\lambda \neq 0$. Choose $f \in \mathrm{Lip}_0(X)$ such that $f(x) = 1$, so that

$$\lambda = \omega(f) = \omega(f^2) = \omega(f)^2 = \lambda^2.$$

This yields $\lambda = 1$, i.e. $\omega = \omega_x := (f \mapsto f(x))$ □

Spectra. In these lecture notes we started with **unital** algebras (which we simply called "algebras"). At the present, we have encountered **non-unital** algebras, e.g. $\mathrm{Lip}_0(X)$ and its ideals on a compact metric space X. In the sequel, let the word "algebra" stand for both unital and non-unital algebras. We say that a *homomorphism* is a linear multiplicative mapping between algebras such that if both algebras are unital then one unit element is mapped to another; the set of homomorphisms $\mathcal{A} \to \mathcal{B}$ is denoted by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Notice that $0 \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ if and only if $\mathcal{A}$ or $\mathcal{B}$ is non-unital. With these nominations, let the *spectrum* of a Banach space and a commutative topological algebra $\mathcal{A}$ be

$$\mathrm{Spec}(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}, \mathbb{K}).$$

If furthermore $\mathcal{A} \cong E'$ for a Banach space E, let

$$\mathrm{Spec}^{w^*}(\mathcal{A}) := \{\omega \in \mathrm{Spec}(\mathcal{A}) \mid \omega \text{ is weak}^*\text{–continuous}\}.$$

Endow all these spectra with the metric given by the norm of the Banach space $\mathcal{A}'$; there are also the relative weak*-topologies of $\mathcal{A}'$ on the spectra.

Theorem. *Let X be a compact metric space. Then the metric topology and the relative weak*-topology of* $\mathrm{Spec}^{w^*}(\mathrm{Lip}_0(X))$ *are the same, and X is isometric to* $\mathrm{Spec}^{w^*}(\mathrm{Lip}_0(X))$. *Moreover,* $\mathrm{Spec}^{w^*}(\mathrm{Lip}_0(X)) = \mathrm{Spec}(\mathrm{Lip}_0(X))$.

Proof. Let us denote $\mathcal{A} := \mathrm{Lip}_0(X)$. The weak*-topology on $K := \mathrm{Spec}^{w^*}(\mathcal{A})$ is the topology induced by the family $\{\widehat{f} \mid f \in \mathcal{A}\}$, where $\widehat{f} : K \to \mathbb{K}$ is defined by $\widehat{f}(\omega) := \omega(f)$ (sort of Gelfand transform).

The previous Corollary indicates that K is the set of evaluation homomorphisms $\omega_x = (f \mapsto f(x))$, and we know that

$$\iota = (x \mapsto m_{xe} = \omega_x) : X \mapsto AE(X) \subset \mathcal{A}'$$

is an isometry. Hence X is isometric to K.

The norm topology of $\mathcal{A}'$ is stronger than the weak*-topology, so that the metric topology on K is stronger than the relative weak*-topology. Notice that $f_y \in \mathcal{A}$, where $\widehat{f_y}(\omega_x) = f_y(x) = d(x,y) - d(e,y)$; hence $\widehat{f_y} : K \to \mathbb{R}$ is weak*-continuous on K, so that $\widehat{f_y}^{-1}(U) \subset K$ is weak*-open for every open set $U \subset \mathbb{R}$. Thus the metric ball

$$\begin{aligned} B(\omega_y, \varepsilon) &= \{\omega_x : \|\omega_x - \omega_y\| < \varepsilon\} = \{\omega_x : d(x,y) < \varepsilon\} \\ &= \{\omega_x : \widehat{f_y}(\omega_x) < \varepsilon - d(e,y)\} \end{aligned}$$

is a weak*-open set. Clearly $\{B(\omega_y,\varepsilon) \mid y \in X,\ \varepsilon > 0\}$ is a basis for the metric topology of the spectrum; thereby the metric topology is weaker than the weak*-topology. Consequently, the topologies must be the same.

Let us extend $\omega \in \mathrm{Spec}(\mathcal{A}) \subset \mathcal{A}'$ linearly to $\widetilde{\omega} : \mathrm{Lip}(X) \to \mathbb{K}$ by setting $\widetilde{\omega}(\mathbb{I}) = 1$. Then $\widetilde{\omega} \in \mathrm{Spec}(\mathrm{Lip}(X))$. **Assume** that for every $x \in X$ there exists $f_x \in \mathrm{Ker}(\widetilde{\omega})$ such that $f_x(x) \neq 0$. Then pick a neighborhood $U_x \in \mathcal{V}(x)$ such that $0 \notin f_x(U_x)$. Due to the compactness of X we may pick a finite subcover $\{U_{x_j}\}_{j=1}^n$ out of the open cover $\{U_x \mid x \in X\}$. Then

$$f := \sum_{j=1}^n |f_{x_j}|^2 = \sum_{j=1}^n \overline{f_{x_j}}\ f_{x_j} \in \mathrm{Ker}(\widetilde{\omega});$$

so f belongs to an ideal of $\mathrm{Lip}(X)$, but on the other hand $f(x) > 0$ for every $x \in X$, so that $1/f \in \mathrm{Lip}(X)$ as the reader may verify — a **contradiction**. Hence there exists $x \in X$ such that $f(x) = 0$ for every $f \in \mathrm{Ker}(\widetilde{\omega})$. The reader may prove analogies of the Lipschitz Stone–Weierstrass Theorem and its consequences replacing (non-unital) subalgebras of $\mathrm{Lip}_0(X)$ by (unital) subalgebras of $\mathrm{Lip}(X)$; of course, the weak*-convergence has to be replaced by the pointwise convergence of bounded nets; then it follows that $\mathrm{Ker}(\widetilde{\omega}) = \{f \in \mathrm{Lip}(X) \mid f(x) = 0\}$, which would imply that $\widetilde{\omega} = (f \mapsto f(x))$.

Hence $\widetilde{\omega} = (f \mapsto f(x))$ for some $x \in X$, and consequently $\omega = \omega_x$. Evaluation homomorphisms are weak*-continuous, so that we have proven that $\mathrm{Spec}(\mathcal{A}) = K$ □

Theorem. *Let $\mathcal{A}$ be a Banach space and a non-unital commutative topological algebra, and endow* $\mathrm{Spec}(\mathcal{A})$ *with the relative metric of $\mathcal{A}'$. Then* $\mathrm{Spec}(\mathcal{A})$ *is a complete pointed metric space of finite diameter, and the extended Gelfand transform*

$$(f \mapsto \widehat{f}) : \mathcal{A} \to \mathrm{Lip}_0(\mathrm{Spec}(\mathcal{A})),$$

(where $\widehat{f}(\omega) := \omega(f)$ for $f \in \mathcal{A}$ and $\omega \in \mathrm{Spec}(\mathcal{A})$) is of norm ≤ 1.

Proof. We may always endow $\mathcal{A}$ with an equivalent Banach algebra norm (even though the algebra is non-unital). From the Gelfand theory of commutative Banach algebras, we know that $\mathrm{Spec}(\mathcal{A})$ is a bounded weak*-closed (even weak*-compact) subset of $\mathcal{A}'$; hence the metric is complete, and the diameter is finite.

Now let $x \mapsto \|x\|$ be the original norm of $\mathcal{A}$. Let $\phi, \psi \in \mathrm{Spec}(\mathcal{A})$. Then

$$|\widehat{x}(\phi) - \widehat{x}(\psi)| = |(\phi - \psi)(x)| \leq \|\phi - \psi\|\ \|x\|,$$

so that $L(\widehat{x}) \leq \|x\|$. Notice that $\widehat{x}(0) = 0$, so that the proof is complete □

Theorem. *Let $\mathcal{A}$ be a commutative Banach algebra, and endow* $\mathrm{Spec}(\mathcal{A})$ *with the relative metric of $\mathcal{A}'$. Then* $\mathrm{Spec}(\mathcal{A})$ *is a complete metric space of diameter at most* 2*, and the Gelfand transform* $(f \mapsto \widehat{f}) : \mathcal{A} \to \mathrm{Lip}(\mathrm{Spec}(\mathcal{A}))$ *is of norm* 1*.*

Proof. In the Gelfand theory we have seen that $\mathrm{Spec}(\mathcal{A})$ belongs to the closed unit ball of $\mathcal{A}'$, so that the diameter of the spectrum is at most 2. If $\phi \in \mathrm{Spec}(\mathcal{A})$ and $x \in \mathcal{A}$ then $|\widehat{x}(\phi)| = |\phi(x)| \leq \|x\|$, and the rest of the proof is as in the previous Theorem $\square$

Remark. Let $\mathcal{A}$ be a Banach space and a non-unital topological algebra. If $\mathrm{Spec}(\mathcal{A})$ is compact in the metric topology then the metric topology is the relative weak*-topology, and $\mathrm{Lip}_0(\mathrm{Spec}(\mathcal{A})) \subset C(\mathrm{Spec}(\mathcal{A}))$.