11 Metrizability

Next we try to construct metrics on compact spaces. We shall learn that a compact space is metrizable if and only if the corresponding commutative C*-algebra is separable. Metrizability is equivalent to the existence of a countable family of continuous functions separating the points of the space. As a vague analogy to the manifolds, the reader may view such a countable family as a set of coordinate functions on the space.

Theorem. *If $\mathcal{F} \subset C(X)$ is a countable family separating the points of a compact space (X, τ) then X is metrizable.*

Proof. Let $\mathcal{F} = \{f_n\}_{n=0}^{\infty} \subset C(X)$ separate the points of X. We can assume that $\|f_n\| \leq 1$ for every $n \in \mathbb{N}$; otherwise consider for instance functions $x \mapsto f_n(x)/(1 + |f_n(x)|)$. Let us define

$$d(x, y) := \sup_{n \in \mathbb{N}} 2^{-n} |f_n(x) - f_n(y)|$$

for every $x, y \in X$. Next we prove that $d : X \times X \to [0, \infty[$ is a metric: $d(x, y) = 0 \Leftrightarrow x = y$, because $\{f_n\}_{n=0}^{\infty}$ is a separating family. Clearly also $d(x, y) = d(y, x)$ for every $x, y \in X$. Let $x, y, z \in X$. We have the triangle inequality:

$$\begin{aligned} d(x, z) &= \sup_{n \in \mathbb{N}} 2^{-n} |f_n(x) - f_n(z)| \\ &\leq \sup_{n \in \mathbb{N}} (2^{-n} |f_n(x) - f_n(y)| + 2^{-n} |f_n(y) - f_n(z)|) \\ &\leq \sup_{m \in \mathbb{N}} 2^{-m} |f_m(x) - f_m(y)| + \sup_{n \in \mathbb{N}} 2^{-n} |f_n(y) - f_n(z)| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Hence d is a metric on X.

Finally, let us prove that the metric topology coincides with the original topology, $\tau_d = \tau$: Let $x \in X$, $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Define

$$U_n := f_n^{-1}(\mathbb{D}(f_n(x), \varepsilon)) \in \mathcal{V}_\tau(x), \quad U := \bigcap_{n=0}^{N} U_n \in \mathcal{V}_\tau(x).$$

If $y \in U$ then

$$d(x, y) = \sup_{n \in \mathbb{N}} 2^{-n} |f_n(x) - f_n(y)| < \varepsilon.$$

Thus $x \in U \subset B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. This proves that the original topology τ is finer than the metric topology τ_d, i.e. $\tau_d \subset \tau$. Combined with the facts that (X, τ) is compact and (X, τ_d) is Hausdorff, this implies that we must have $\tau_d = \tau$ $\square$

Corollary. *Let X be a compact Hausdorff space. Then X is metrizable if and only if it has a countable basis.*

Proof. Suppose X is a compact space, metrizable with a metric d. Let $r > 0$. Then $\mathcal{B}_r = \{B_d(x, r) \mid x \in X\}$ is an open cover of X, thus having a finite subcover $\mathcal{B}'_r \subset \mathcal{B}_r$. Then $\mathcal{B} := \bigcup_{n=1}^{\infty} \mathcal{B}'_{1/n}$ is a countable basis for X.

Conversely, suppose X is a compact Hausdorff space with a countable basis $\mathcal{B}$. Then the family

$$\mathcal{C} := \{(B_1, B_2) \in \mathcal{B} \times \mathcal{B} \mid \overline{B_1} \subset B_2\}$$

is countable. For each $(B_1, B_2) \in \mathcal{C}$ Urysohn's Lemma provides a function $f_{B_1 B_2} \in C(X)$ satisfying

$$f_{B_1 B_2}(\overline{B_1}) = \{0\} \quad \text{and} \quad f_{B_1 B_2}(X \setminus B_2) = \{1\}.$$

Next we show that the countable family

$$\mathcal{F} = \{f_{B_1 B_2} : \ (B_1, B_2) \in \mathcal{C}\} \subset C(X)$$

separates the points of X: Take $x, y \in X$, $x \neq y$. Then $W := X \setminus \{y\} \in \mathcal{V}(x)$. Since X is a compact Hausdorff space, there exists $U \in \mathcal{V}(x)$ such that $\overline{U} \subset W$. Take $B', B \in \mathcal{B}$ such that $x \in B' \subset \overline{B'} \subset B \subset U$. Then $f_{B'B}(x) = 0 \neq 1 = f_{B'B}(y)$. Thus X is metrizable $\square$

Conclusion. *Let X be a compact Hausdorff space. Then X is metrizable if and only if $C(X)$ is separable (i.e. contains a countable dense subset).*

Proof. Suppose X is a metrizable compact space. Let $\mathcal{F} \subset C(X)$ be a countable family separating the points of X (as in the proof of the previous Corollary). Let $\mathcal{G}$ be the set of finite products of functions f for which $f \in \mathcal{F} \cup \mathcal{F}^* \cup \{\mathbb{I}\}$; the set $\mathcal{G} = \{g_j\}_{j=0}^{\infty}$ is countable. The linear span $\mathcal{A}$ of $\mathcal{G}$ is the involutive algebra generated by $\mathcal{F}$ (the smallest $*$-algebra containing $\mathcal{F}$); due to the Stone–Weierstrass Theorem, $\mathcal{A}$ is dense in $C(X)$. If $S \subset \mathbb{C}$ is a countable dense set then

$$\{\lambda_0 \mathbb{I} + \sum_{j=1}^{n} \lambda_j g_j \mid n \in \mathbb{Z}^+, \ (\lambda_j)_{j=0}^{n} \subset S\}$$

is a countable dense subset of $\mathcal{A}$, thereby dense in C(X).

Conversely, assume that $\mathcal{F} = \{f_n\}_{n=0}^{\infty} \subset C(X)$ is a dense subset. Take $x, y \in X$, $x \neq y$. By Urysohn's Lemma there exists $f \in C(X)$ such that $f(x) = 0 \neq 1 = f(y)$. Take $f_n \in \mathcal{F}$ such that $\|f - f_n\| < 1/2$. Then

$$|f_n(x)| < 1/2 \quad \text{and} \quad |f_n(y)| > 1/2,$$

so that $f_n(x) \neq f_n(y)$; $\mathcal{F}$ separates the points of X $\square$

Exercise*. Prove that a topological space with a countable basis is separable. Prove that a metric space has a countable basis if and only if it is separable.

Exercise. There are non-metrizable separable compact Hausdorff spaces! Prove that X is such a space, where

$$X = \{f : [0,1] \to [0,1] \mid x \leq y \Rightarrow f(x) \leq f(y)\}$$

is endowed with a relative topology. Hint: Tihonov's Theorem.