1 Informal introduction

These lecture notes present a fundamental relationship between topology, measure theory and algebra. Briefly, if we want to study properties of a space X, we may alternatively examine some algebra of functions on X. With suitable topological restrictions, there will be a bijective correspondence between spaces and algebras (equivalence of categories, if you insist). Topology and measure theory of X can then be stated in the terms of a topological function algebra. And it will turn out that the tools that are developed for the study of function algebras work as well for non-commutative algebras.

Let us begin with a trivial example. Let X be a finite set. Let $\mathcal{A} = \mathcal{F}(X)$ be the set of the complex-valued functions $f : X \to \mathbb{C}$. Then $\mathcal{A}$ is naturally a $\mathbb{C}$-vector space:

$$(f+g)(x) := f(x) + g(x), \qquad (\lambda f)(x) := \lambda\ f(x)$$

for every $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Moreover, $\mathcal{A}$ is an algebra when endowed with the product

$$(fg)(x) := f(x)\ g(x)$$

and with the unit element $\mathbb{I} = \mathbb{I}_{\mathcal{A}}$, which is the constant function $\mathbb{I}(x) \equiv 1$. Let $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ denote the set of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$. When $x \in X$, the evaluation mapping

$$(f \mapsto f(x)) : \mathcal{A} \to \mathbb{C}$$

is a homomorphism. Hence we may think that X is a subset of $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$. Actually, it turns out that the evaluation mappings are the only homomorphisms $\mathcal{A} \to \mathbb{C}$. Therefore we may even claim

$$X = \mathrm{Hom}(\mathcal{A}, \mathbb{C}).$$

It can be proven that every isomorphism $\mathcal{A} \to \mathcal{A}$ arises from a bijection $X \to X$. And if $\emptyset \neq I \subset X$, it is easy to see that

$$\mathcal{I} := \{f \in \mathcal{A} \mid \forall x \in I : f(x) = 0\}$$

is an ideal of the algebra $\mathcal{A}$, and that there are no other ideals; thus the non-empty subsets of X are in bijective correspondence with the ideals of $\mathcal{A}$. Any $\lambda : X \to \mathbb{C}$ defines a linear functional $\Lambda : \mathcal{A} \to \mathbb{C}$ by

$$\Lambda f := \sum_{x \in X} f(x)\ \lambda(x),$$

which can be thought as an integral with respect to the λ-weighted counting measure on X; conversely, any linear functional on $\mathcal{A}$ arises this way.

From analysis point of view, the discrete topology is the most reasonable topology for a finite set X, and the counting measure is the natural choice for measure theory. We should not endow an infinite set with the discrete topology nor with the counting measure. Instead, non-trivial topology and measure theory will be necessary. The framework is the theory of commutative C*-algebras ("C-star-algebra"), an extremely beautiful branch of functional analysis. In essence, this theory boils down to the following:

Theorem. *Compact Hausdorff spaces X and Y are homeomorphic if and only if the function algebras $C(X)$ and $C(Y)$ of complex-valued continuous functions are isomorphic.*

Thus the topological and measure theoretic information of some topological space X is equivalent to the topologic-algebraic information of $C(X)$. The same phenomenon occurs also for differentiability properties. We may study directly the geometry of a space, but as well we may study algebras of functions on it! This is called **"commutative geometry"**, as function algebras are commutative. Now this remark almost forces us to generalize: We may study certain non-commutative algebras using similar tools as in the commutative case. Hence the name **"non-commutative geometry"**.

The reader may wonder why these themes should be relevant. We have already expressed the nice connections between different branches of mathematics. Let us go back in the history: In 1925, Werner Heisenberg (1901-1976) and Erwin Schrödinger (1887-1961) initiated the quantum mechanics. Heisenberg applied matrix algebras, while Schrödinger practically studied Fourier analysis, but their theories were essentially equivalent. However, a precise mathematical foundation for quantum physics was lacking. This was the main reason for János von Neumann (1903-1957) to develop the Hilbert spaces and the spectral theory of normal operators in 1929-1930. In this context, a quantum mechanical system is presented as a partial differential equation (*Schrödinger equation*) on a Hilbert space $\mathcal{H}$, where unit vectors $\phi \in \mathcal{H}$ ($\|\phi\| = 1$) are *states* of the system. The measurable quantities, or *observables*, of the system are the self-adjoint linear operators ($A^* = A$) on $\mathcal{H}$. Also the unbounded operators are interesting, for instance the location and momentum operators on $L^2(\mathbb{R}^n)$. When we measure a quantity, the result is not the full information about the observable but merely a value from the spectrum of the operator (e.g. try to locate a particle in the space). The interesting thing is then to find a spectral decomposition of an observable, analogous to the diagonalization of a Hermitian matrix.

Well, this is not a physical Theory of Everything. Anyhow, $\mathcal{L}(\mathcal{H})$ is the natural first stage in developing the operator algebras. In 1936, the next ingenious step was the theory of von Neumann algebras, capturing some measure theoretic properties of classical L^∞-type spaces. These algebras were a special case of C*-algebras, whose theory emerged in the early of 1940s mainly by Israil Gelfand (1913-).

Practical definition. *A* C*-algebra $\mathcal{A}$ *is a norm closed involutive subalgebra of* $\mathcal{L}(\mathcal{H})$ *for some Hilbert space* $\mathcal{H}$.

Equivalent abstract definition. *A* C*-*algebra* $\mathcal{A}$ *is a Banach space and a* $\mathbb{C}$-*algebra with involution* $x \mapsto x^*$ *such that*

$$\|xy\| \leq \|x\| \, \|y\|, \quad \|\mathbb{I}_{\mathcal{A}}\| = 1, \quad \|x^* x\| = \|x\|^2$$

for every $x, y \in \mathcal{A}$.

Gelfand's idea was to look at a "mirror reflection" of a commutative algebra. Actually, this approach can be dated back at least to Hilbert's *Nullstellensatz* in algebraic geometry, in 1893. Let $\mathcal{A}$ be a commutative C*-algebra. Let $X = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ be the set of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$. The *Gelfand transform* of an element $f \in \mathcal{A}$ is the function $\hat{f} : X \to \mathbb{C}$ defined by

$$\hat{f}(x) := x(f),$$

where $x \in X = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$. This seems astonishingly simple, but is fundamental. Now Gelfand proved that X is a compact Hausdorff space in the natural topology inherited from the weak*-topology of the dual space $\mathcal{A}' = \mathcal{L}(\mathcal{A}, \mathbb{C})$. Moreover:

Gelfand–Naimark Theorem (1943). *Any commutative C*-algebra is isometrically isomorphic to the algebra* $C(X)$ *for some compact Hausdorff space* X.

Gelfand–Naimark Theorem is the starting point of the non-commutative geometry, which was initiated by Alain Connes in the 1980s. By now, this huge subject contains such topics as Hopf algebras and quantum groups, K-theory for operator algebras, non-commutative integrodifferential calculus, non-commutative manifolds, and so on. This machinery has been applied e.g. in particle physics, quantum field theory and string theory.

However, within the limited time we have, we only present some of the early fundamental results of topology and operator algebras. This hopefully provides a solid background for the reader to investigate non-commutative geometry further.

To distill some of the essential results that will be obtained in these lecture notes, we present a "dictionary" relating topology, measure theory and algebra; here X is a compact Hausdorff space:

Topology / Measure theory	$\leftrightarrow$	**Algebra**
$C(X)$	$\leftrightarrow$	commutative C*-algebra $\mathcal{A}$
homeomorphism $X \to X$	$\leftrightarrow$	isomorphism $\mathcal{A} \to \mathcal{A}$
point $\in X$	$\leftrightarrow$	maximal ideal $\in \mathcal{A}$ or homomorphism $\mathcal{A} \to \mathbb{C}$
non-empty closed subset $\subset X$	$\leftrightarrow$	closed ideal $\subset \mathcal{A}$ or quotient algebra $\mathcal{A}$/ideal
range of $f \in C(X)$	$\leftrightarrow$	spectrum of an element $\in \mathcal{A}$
X metrizable	$\leftrightarrow$	$\mathcal{A}$ separable
X disconnected	$\leftrightarrow$	$\exists f \in \mathcal{A}: \; f^2 = f, 0 \neq f \neq 1$
complex measure on X	$\leftrightarrow$	bounded linear functional $\mathcal{A} \to \mathbb{C}$
positive measure on X	$\leftrightarrow$	positive linear functional $\mathcal{A} \to \mathbb{C}$
$\vdots$	$\leftrightarrow$	$\vdots$