TKK Matematiikan laitos Arponen/Pursiainen

**Mat-1.3651 Numerical Linear Algebra**, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

**Exercise 10** (10.4.2008)

Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise. This is the last exercise. **You may take the course exam on Thursday 17th April, 2008 at 2.30pm in lecture hall G.** In the April exam 25% of the grade is determined by the hand-in home assignments.
**Another possibility to take the exam is on Saturday 10th May, 2008.** Check the official exam schedules for the correct location. If you take the May exam, you must specify whether or not your home assignments affect the grade.

* 1. Suppose that in the Arnoldi iteration (given $A$ and $b$) an entry $h_{n+1,n} = 0$ is encountered.

(a) Show how the formula $AQ_n = Q_{n+1}\tilde{H}_n$ can now be simplified. What does this imply about the structure of a full $m \times m$ Hessenberg reduction $A = QHQ^*$ of $A$?

(b) Show that $\mathcal{K}_n$ is an invariant subspace of $A$.

(c) Show that the Krylov subspaces fulfil $\mathcal{K}_n = \mathcal{K}_{n+1} = \mathcal{K}_{n+2} = \dots$ .

(d) Show that $\Lambda(H_n) \subset \Lambda(A)$, i.e. each eigenvalue of $H_n$ is an eigenvalue of $A$.

(e) Show that if $A$ is nonsingular, then the solution $x$ to the system $Ax = b$ lies in $\mathcal{K}_n$.

2. Let $A \in \mathbb{C}^{m\times m}$ and $b \in \mathbb{C}^m$ be given, and $K_n$ the related $m \times n$ Krylov matrix with $K_n^+$ the pseudoinverse. Let $p_n$ be the characteristic polynomial of the Hessenberg $H_n$ in the Arnoldi iteration. How is $p_n$ seen in the matrix $K_n^+ AK_n$?

* 3. Let $A = \begin{pmatrix} 0 & 1 \\ I_{m-1} & 0 \end{pmatrix} \in \mathbb{R}^{m\times m}$ and $b = e_1 \in \mathbb{R}^m$. Compute the Ritz values.

4. GMRES has the least squares problem

$$\|\tilde{H}_n y - \|b\| e_1\| = \min$$

to solve.

(a) Describe an $O(n^2)$ algorithm based on QR factorization by Givens rotations (see Exercise 5).

(b) Show how the operation count can be improved to $O(n)$, as mentioned in the lecture, if the problem for step $n-1$ has already been solved.

* 5. We know that CG is (also) an iterative minimization of the function $\varphi(x) = \frac{1}{2}x^T Ax - x^T b$, $x \in \mathbb{R}^m$. Another way to minimize the same function is the method of *steepest descent*.

(a) Derive the formula $\nabla\varphi(x) = -r$. The steepest descent iteration corresponds to the choice $p_n = r_n$ instead of $p_n = r_n + \beta_n p_{n-1}$ in CG.

(b) Determine the formula for the optimal step length $\alpha_n$ of the steepest descent iteration.

(c) Write down the full steepest descent iteration. There are three operations inside the main loop.