Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 4 (14.2.2008)

These are held in the computer classroom Y339b (close to Y313). Please hand in the exercise marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise. In addition to that, hand in the exercises marked with [Comp. hand-in] in the *next* exercise session (21st February, that is).

1. [Comp. hand-in] (hand in your answers to (d),(e), and (f).) Here you will study compression of information by the SVD. Load and draw the following picture in Matlab:

```
load clown.mat;     % this is one of the demo files
whos;               % see what you loaded
image(X);           % interprete the matrix X as an image
colormap(map);      % fix the colours (try also colormap('grey');)
```

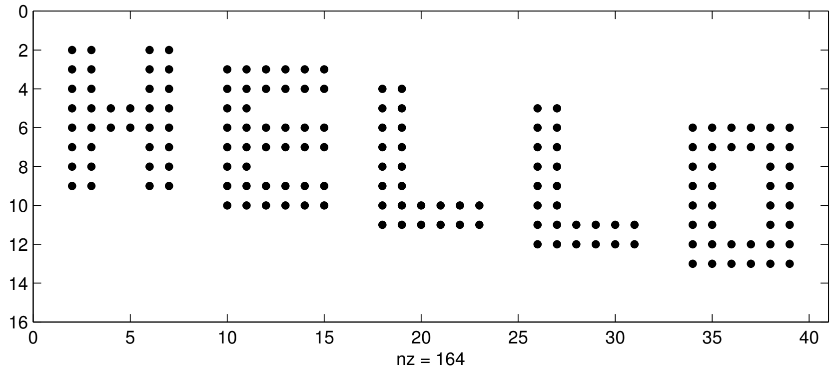
   (a) Compute the SVD of X.
   (b) Make the best "rank-k" approximations $X_k := \sum_{j=1}^{k} \sigma_j u_j v_j^*$ for some k, at least $k \in \{3, 7, 10, 20\}$.
   (c) Plot the approximations.
   (d) The storage: how much data is needed to construct X_k, compared to the amount of data in the original X?
   (e) By looking at all singular values (plot them on a suitable scale), give your opinion on what would be a suitable k? There is no optimal value, but consider the "visual quality"/storage value by your own eyes.
   (f) Assume you have sent over a network the data representing X_k. How would you update that approximation to X_{k+10}?

2. Write a Matlab program which, given a real 2×2 matrix A, plots the right singular vectors v_1 and v_2 in the unit circle and also the left singular vectors u_1 and u_2 in the appropriate ellipse. Apply your program to the 2×2 matrices of Exercise 2, question 4:

$$\text{(a)} \quad \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \text{(b)} \quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{(c)} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

3. [Comp. hand-in] Write a Matlab function `[Q,R]=mgs(A)` that computes a reduced QR factorization $A = \hat{Q}\hat{R}$ of an $m \times n$ matrix A with $m \geq n$ using modified Gram-Schmidt orthogonalization. The output variables are a matrix $Q \in \mathbb{C}^{m \times n}$ with orthonormal columns and a triangular matrix $R \in \mathbb{C}^{n \times n}$.

4. (a) Write a Matlab program that sets up a 15×40 matrix with entries 0 everywhere except for the values 1 in the positions indicated in the picture below. The upper-leftmost 1 is in position $(2, 2)$, and the lower-rightmost 1 is in position $(13, 39)$. This picture was produced with the command `spy(A)`.



   (b) Call `svd` to compute the singular values of A. Plot these numbers using both `plot` and `semilogy`. What is the exact rank of A? Can you see this in the computed singular values?

   (c) For $i = 1$ to $\text{rank}(A)$, construct the rank-i matrix B that is the best approximation to A in the 2-norm. Use the command `pcolor(B)` with `colormap(gray)` to create images of these various approximations.

5. (a) Write a Matlab function `[W,R]=house(A)` that computes an implicit representation of a full QR factorization $A = QR$ of an $m \times n$ matrix A with $m \geq n$ using Householder reflections. The output variables are a lower-triangular matrix $W \in \mathbb{C}^{m \times n}$ whose columns are the vectors v_k defining the successive Householder reflections, and a triangular matrix $R \in \mathbb{C}^{n \times n}$.

   (b) Write a Matlab function `Q=formQ(W)` that takes the matrix W produced by `house` as input and generates a corresponding $m \times m$ orthogonal matrix Q.

6. [Comp. hand-in] Let Z be the matrix

$$Z = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{pmatrix}.$$

Compute three reduced QR factorizations of Z in Matlab: by the Gram-Schmidt routine `mgs` of Exercise 3, by the Householder routines `house` and `formQ` of Exercise 5, and by Matlab's built-in command `[Q,R]=qr(Z,0)`. Compare these three and comment on any differences you see.