Chapter 5

Wavelet transforms and frames

1. The continuous wavelet transform

In this section we study a version of the so-called continuous wavelet transform applied to the ridge functions appearing in a neural network with one hidden layer.

If now ψ is a given function, then we define

$$\psi_{\mathbf{u},a,b}(\mathbf{x}) = \frac{1}{\sqrt{a}}\psi\left(\frac{\mathbf{u}\cdot\mathbf{x}-b}{a}\right), \quad |\mathbf{u}|=1, \quad a>0, \quad b\in\mathbb{R}.$$

We have the following result.

Theorem 18. *Let $d \geq 1$ and let ψ and $\varphi \in L^1(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)||\hat{\varphi}(\omega)|}{|\omega|^d}\,\mathrm{d}\omega < \infty \quad \textit{and} \quad K_{\psi,\varphi} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{\overline{\hat{\psi}(\omega)}\hat{\varphi}(\omega)}{|\omega|^d}\,\mathrm{d}\omega \neq 0.$$

If $f \in L^1(\mathbb{R}^d)$ is such that $\hat{f} \in L^1(\mathbb{R}^d)$, then

$$f(\mathbf{x}) = \frac{1}{K_{\psi,\varphi}} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{\mathbb{R}} \langle f, \psi_{\mathbf{u},a,b}\rangle\, \varphi_{\mathbf{u},a,b}(\mathbf{x})\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}\mathbf{u}.$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$.

Observe that if ψ and φ are real-valued functions, then $K_{\psi,\varphi}$ is real-valued as well. From the proof we see that we have

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty \left| \int_{\mathbb{R}} \langle f, \psi_{\mathbf{u},a,b}\rangle\, \varphi_{\mathbf{u},a,b}(\mathbf{x})\,\mathrm{d}b \right| \mathrm{d}a\,\mathrm{d}\mathbf{u} < \infty,$$

and that the integral $\int_{\mathbb{R}} \langle f, \psi_{\mathbf{u},a,b}\rangle\, \varphi_{\mathbf{u},a,b}(\mathbf{x})\,\mathrm{d}b$ is the convolution of L^1-functions, and hence well-defined. If $\psi = \varphi$, then it is not difficult to show that the triple integral converges absolutely as well.

Proof. Let $\mathbf{u} \in \mathbb{R}^d$ be such that $|\mathbf{u}| = 1$. We define the Radon-transform $P_{\mathbf{u}} f$ as follows:

$$(P_{\mathbf{u}} f)(t) = \int_{\mathbb{R}^{d-1}} f(t\mathbf{u} + U^{\perp}\mathbf{s})\,\mathrm{d}\mathbf{s},$$

where $U^{\perp}$ is a $d \times (d-1)$ matrix with columns that form an orthonormal basis for the subspace of vectors in $\mathbb{R}^d$ orthogonal to $\mathbf{u}$. It is not difficult to show that $P_{\mathbf{u}} f \in L^1(\mathbb{R})$ and that

$$\widehat{P_{\mathbf{u}} f}(\underline{\omega}) = \hat{f}(\underline{\omega}\mathbf{u}). \tag{54}$$

Furthermore, we let, abusing our notation somewhat,

$$\psi_a(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t}{a}\right) \quad \text{and} \quad \tilde{\psi}_a(t) = \overline{\psi_a(-t)}, \quad a > 0 \quad t \in \mathbb{R}.$$

We observe that

$$\langle f, \psi_{\mathbf{u},a,b}\rangle = (\tilde{\psi}_a * P_{\mathbf{u}} f)(b). \tag{55}$$

We let

$$\phi(\underline{\omega}) = \overline{\hat{\psi}(\underline{\omega})}\hat{\varphi}(\underline{\omega}) + \overline{\hat{\psi}(-\underline{\omega})}\hat{\varphi}(-\underline{\omega}),$$

and observe that

$$\int_0^\infty \phi(a\omega)\frac{1}{a^d}\,\mathrm{d}a = \omega^{d-1}\int_0^\infty \phi(a)\frac{1}{a^d}\,\mathrm{d}a$$
$$= \omega^{d-1}\int_{\mathbb{R}} \frac{\overline{\hat{\psi}(\eta)}\hat{\varphi}(\eta)}{|\eta|^d}\,\mathrm{d}\eta = \omega^{d-1}K_{\psi,\varphi}, \quad \omega > 0.$$

The same calculation shows, of course, that there is a constant C such that

$$\int_0^\infty |\phi(a\omega)|\,\frac{1}{a^d}\,\mathrm{d}a \le C\omega^{d-1}, \quad \omega > 0. \tag{56}$$

If we now let $\mathbf{x} \in \mathbb{R}^d$ be arbitrary and define

$$g(\mathbf{x}) \overset{\text{def}}{=} \int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}2\pi\omega\mathbf{u}\cdot\mathbf{x}}\phi(a\omega)\hat{f}(\omega\mathbf{u})\frac{1}{a^d}\,\mathrm{d}a\,\mathrm{d}\omega\,\mathrm{d}\mathbf{u}, \tag{57}$$

then it follows from (56) and our assumptions on f that this integral converges absolutely, and we have in fact

$$g(\mathbf{x}) = K_{\psi,\varphi}\int_{\mathbb{S}^{d-1}}\int_0^\infty \mathrm{e}^{\mathrm{i}2\pi\omega\mathbf{u}\cdot\mathbf{x}}\hat{f}(\omega\mathbf{u})\omega^{d-1}\,\mathrm{d}\omega\,\mathrm{d}\mathbf{u} \tag{58}$$
$$= K_{\psi,\varphi}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}2\pi\mathbf{y}\cdot\mathbf{x}}\hat{f}(\mathbf{y})\,\mathrm{d}\mathbf{y} = K_{\psi,\varphi}f(\mathbf{x}).$$

By Fubini's theorem and the fact that $\mathbb{S}^{d-1}$ is invariant under the mapping $\mathbf{u} \mapsto -\mathbf{u}$ we get

$$(59)\quad g(\mathbf{x}) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty \Big(e^{i2\pi\omega \mathbf{u}\cdot\mathbf{x}} \overline{\hat{\psi}(a\omega)} \hat{\varphi}(a\omega) \hat{f}(\omega\mathbf{u}) + e^{-i2\pi\omega \mathbf{u}\cdot\mathbf{x}} \overline{\hat{\psi}(-a\omega)} \hat{\varphi}(-a\omega) \hat{f}(\omega\mathbf{u}) \Big) \frac{1}{a^d}\, d\omega\, da\, d\mathbf{u}$$
$$= \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_{\mathbb{R}} e^{i2\pi\omega \mathbf{u}\cdot\mathbf{x}} \overline{\hat{\psi}(a\omega)} \hat{\varphi}(a\omega) \hat{f}(\omega\mathbf{u}) \frac{1}{a^d}\, d\omega\, da\, d\mathbf{u}.$$

Next we note from (54) that the Fourier transform of the function $\tilde{\psi}_a * P_{\mathbf{u}} * \varphi_a$ is $a\overline{\hat{\psi}(a\underline{\omega})}\hat{\varphi}(a\underline{\omega})\hat{f}(\underline{\omega}\mathbf{u})$, and therefore we get by the Fourier inversion formula

$$(60)\qquad g(\mathbf{x}) = \int_{\mathbb{S}^{d-1}} \int_0^\infty (\tilde{\psi}_a * P_{\mathbf{u}} f * \varphi_a)(\mathbf{u}\cdot\mathbf{x}) \frac{1}{a^{d+1}}\, da\, d\mathbf{u}.$$

(By the results above we know that $\int_{\mathbb{S}^{d-1}} \int_0^\infty |(\tilde{\psi}_a * P_{\mathbf{u}} f * \varphi_a)(\mathbf{u}\cdot\mathbf{x})| \frac{1}{a^{d+1}}\, da\, d\mathbf{u} < \infty$.) Now by (55)

$$(\tilde{\psi}_a * P_{\mathbf{u}} f * \varphi_a)(\mathbf{u}\cdot\mathbf{x}) = \int_{\mathbb{R}} (\tilde{\psi}_a * P_{\mathbf{u}} f)(b) \varphi_a(\mathbf{u}\cdot\mathbf{x} - b)\, db$$
$$= \int_{\mathbb{R}} \langle f, \psi_{\mathbf{u},a,b} \rangle\, \varphi_{\mathbf{u},a,b}(\mathbf{x})\, db.$$

When this result is combined with (58) and (60) we get the claim of the theorem. □

2. Riesz bases and frames

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then a sequence $(e_n)_{n=1}^\infty \subset H$ is an orthonormal basis of H if for all $n,\ m \geq 1$ we have $\langle e_n, e_m \rangle = 0$ if $n \neq m$ and $\|e_n\| = 1$, and the span of the sequence is dense in H.

Theorem 19. *Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $(e_n)_{n=1}\infty \subset H$. Then the following properties are equivalent.*

1. *$(e_n)_{n=1}$ is an orthonormal basis of H.*
2. *$\overline{\operatorname{span}\{e_n\}_{n=1}^\infty} = H$ and*

$$\sum_{n=1}^k |c_n|^2 = \left\| \sum_{n=1}^k c_n e_n \right\|^2,$$

for all numbers $c_1, \ldots, c_k$, $k \geq 1$.

3. $\|e_n\| = 1$, $n \geq 1$ *and*

$$\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^2 = \|f\|^2, \quad f \in H.$$

Next we consider so called Riesz bases, but note that there are many other ways of characterizing such bases than the ones given below.

Theorem 20. *Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $(f_n)_{n=1}^{\infty} \subset H$. Then the following properties are equivalent (and if they hold, the sequence is said to be a Riesz basis):*

(i) *There is an orthonormal basis $(e_n)_{n=1}^{\infty}$ of H and a bounded linear operator $T : H \to H$ with bounded inverse such that $f_n = Te_n$ for each $n \geq 1$.*

(ii) $\overline{\mathrm{span}\{f_n\}_{n=1}^{\infty}} = H$ *and there are positive constants a and b such that and*

$$a \sum_{n=1}^{k} |c_n|^2 \leq \left\| \sum_{n=1}^{k} c_n e_n \right\|^2 \leq b \sum_{n=1}^{k} |c_n|^2,$$

for all numbers $c_1, \dots, c_k$, $k \geq 1$.

(iii) $\overline{\mathrm{span}\{f_n\}_{n=1}^{\infty}} = H$ *and there are positive constants a and B such that*

$$a \sum_{n=1}^{k} |c_n|^2 \leq \|\sum_{n=1}^{k} c_n e_n\|^2,$$

for all numbers $c_1, \dots, c_k$, $k \geq 1$, and

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad f \in H.$$

(iv) $\overline{\mathrm{span}\{f_n\}_{n=1}^{\infty}} = H$ *and there is a sequence $(g_n)_{n=1}^{\infty}$ such that* $\overline{\mathrm{span}\{g_n\}_{n=1}^{\infty}} = H$ *and for all m, $n \geq 1$ we have $\langle f_n, g_m\rangle = 0$ if $n \neq m$ and $\langle f_n, g_n\rangle = 1$, and there is a constant B such that*

$$\left.\begin{aligned} \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \\ \sum_{n=1}^{\infty} |\langle f, g_n\rangle|^2 \leq B\|f\|^2, \end{aligned}\right. \qquad f \in H.$$

(v) *There is a sequence $(g_n)_{n=1}^{\infty}$ such that for all m, $n \geq 1$ we have $\langle f_n, g_m\rangle = 0$ if $n \neq m$ and $\langle f_n, g_n\rangle = 1$, and there are constants*

$0 < A \le B < \infty$

$$\left.\begin{aligned} A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \\ A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, g_n\rangle|^2 \le B\|f\|^2, \end{aligned}\right. \qquad f \in H.$$

Proof. (i)⇒(ii): Since $f_n = Te_n$ for all n we have

$$\sum_{n=1}^{k} c_n f_n = T\left(\sum_{n=1}^{k} c_n e_n\right) \quad \text{and} \quad T^{-1}\sum_{n=1}^{k} c_n f_n = \left(\sum_{n=1}^{k} c_n e_n\right)$$

so that

$$\left\|\sum_{n=1}^{k} c_n f_n\right\|^2 \le \|T\|^2 \left\|\sum_{n=1}^{k} c_n e_n\right\|^2 = \|T\|^2 \sum_{k=1}^{n} |c_n|^2,$$

and

$$\left\|\sum_{n=1}^{k} c_n f_n\right\|^2 \ge \|T^{-1}\|^{-2} \left\|\sum_{n=1}^{k} c_n e_n\right\|^2 = \|T^{-1}\|^{-2} \sum_{k=1}^{n} |c_n|^2.$$

(ii)⇔(iii): Suppose (ii) holds. If c_n, $n = 1, \dots, k$ are arbitrary numbers we have

$$\left|\sum_{n=1}^{k} c_n \langle f, f_n\rangle\right|^2 = \left|\left\langle f, \sum_{n=1}^{k} \overline{c_n} f_n\right\rangle\right|^2 \le \|f\|^2 \left\|\sum_{n=1}^{k} \overline{c_n} f_n\right\|^2 \le b\|f\|^2 \sum_{n=1}^{k} |c_k|^2.$$

If we now choose $c_n = \overline{\langle f, f_n\rangle}$ and let $k \to \infty$, then we get the missing claim.

For the converse we let $f = \sum_{n=1}^{k} c_n f_n$. Then

$$\begin{aligned} \|f\|^4 = |\langle f, f\rangle|^2 = \left|\sum_{n=1} k\overline{c_n} \langle f, f_n\rangle\right|^2 \\ \le \sum_{n=1}^{k} |c_n|^2 \sum_{n=1}^{k} |\langle f, f_n\rangle|^2 \le B\|f\|^2 \sum_{n=1}^{k} |c_n|^2. \end{aligned}$$

When we divide by $\|f\|^2$ we get the desired result.

(iii)⇒(iv): The first inequality implies that for each $m \ge 1$

$$\left\|\sum_{\substack{n=1 \\ n\ne m}}^{k} c_n f_n - f_m\right\| \ge a > 0.$$

Thus $f_m \notin \overline{\operatorname{span}\{\, f_n \mid n \ge 1, n \ne m \,\}}$ and therefore there exists an element $g_m \in H$ such that $\langle f_n, g_m\rangle = 0$ if $n \ne m$ and 1 if $n = m$.

If $f = \sum_{n=1}^{k} c_n f_n$ we must therefore have $c_n = \langle f, g_n \rangle$. Thus we have

$$\sum_{n=1}^{\infty} |\langle f, g_n \rangle|^2 \leq \frac{1}{a} \|f\|^2,$$

for f in a dense subset of H, and by continuity for all $f \in H$. In order to prove that $\overline{\operatorname{span}\{g_n\}_{n=1}^{\infty}} = H$ it suffices to recall that (iii) implies (ii) because then we can conclude that if for some $f \in H$ we have $\langle f, g_n \rangle = 0$ for all $n \geq 1$ then $f = 0$.

(iv)⇒(i): Let $(e_n)_{n=1}^{\infty}$ be an arbitrary orthonormal basis for H. furthermore, Let $f = \sum_{n=1}^{k} c_n f_n$ and $g = \sum_{n=1}^{k} d_n g_n$. By the biorthogonality assumption we have $c_n = \langle f, g_n \rangle$ and $d_n = \langle g, f_n \rangle$. If we now define

$$Sf = \sum_{n=1}^{k} c_n e_n,$$

$$Ug = \sum_{n=1}^{x} k d_n e_n,$$

then we conclude that

$$\|Sf\|^2 = \sum_{n=1}^{k} |c_n|^2 = \sum_{n=1}^{k} |\langle f, g_n \rangle|^2 \leq B \|f\|^2.$$

A similar inequality can be derived for U so that we conclude, since S and U are densely defined that they can be extended to bounded continuous operators on H with norms at most $\sqrt{B}$. The biorthogonality combined with the continuous extension implies that

$$\langle Sf, Ug \rangle = \langle f, g \rangle, \quad f, g \in H.$$

Thus we conclude that

$$\|f\|^2 = \langle f, g \rangle = \langle Sf, Uf \rangle \leq \|Sf\| \|Uf\| \leq \|Sf\| \sqrt{B} \|f\|.$$

Since the range of S is dense in H we conclude that S has a bounded inverse and the proof is completed.

(iv)⇔(v): Assume first that (iv) holds. Since we know that (iv) is equivalent to (i) there is an operator T such that $(T^{-1} f_n)_{n=1}^{\infty}$ is an orthonormal basis. Then

$$\sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 = \sum_{n=1}^{\infty} \left| \langle f, TT^{-1} f_n \rangle \right|^2 = \sum_{n=1}^{\infty} \left| \langle T^* f, T^{-1} f_n \rangle \right|^2 = \|T^* f\|^2 \geq \frac{1}{\|(T^*)^{-1}\|^2} \|f\|^2.$$

Since $(g_n)_{n=1}^{\infty}$ satisfies the same assumptions as $(f_n)_{n=1}^{\infty}$ we get the second conclusion as well.

Suppose next that (v) holds. Then we have only to show that $\overline{\operatorname{span}\{f_n\}_{n=1}^{\infty}} = H$ and $\overline{\operatorname{span}\{g_n\}_{n=1}^{\infty}} = H$ and these claims follow directly because by (v) there cannot be a nonzero vector orthogonal to all vectors f_n or to all vectors g_n. □

By dropping part of the requirements in some of the characterizations one gets so called Bessel sequences and Riesz-Fisher sequences. But it turns out to be very fruitful to formulate a new condition as well.

Definition 21. *Let H be a separable Hilbert space. A sequence $(f_n)_{n=1}^{\infty}$ of elements in H is a frame if there are positive constants A and B (the bounds for the frame) such that*

$$A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2, \quad f \in H.$$

Theorem 22. *If $(f_n)_{n=1}^{\infty}$ is a frame then the formula*

$$Tf = \sum_{n=1}^{\infty} \langle f, f_n\rangle\, f_n. \tag{61}$$

defines a bounded, selfadjoint, invertible, linear operator with $\|T\| \le B$ and $\|T^{-1}\| \le A^{-1}$. Moreover, if $f \in H$, then

$$f = \sum_{n=1} a_n f_n \quad \text{where} \quad a_n = \left\langle T^{-1}f, f_n\right\rangle = \left\langle f, T^{-1}f_n\right\rangle, \quad n \ge 1,$$

and if $f = \sum_{n=1} b_n f_n$, then

$$\sum_{n=1}^{\infty}|b_n|^2 = \sum_{n=1}^{\infty}|a_n|^2 + \sum_{n=1}^{\infty}\left|a_n - b_n\right|^2 \ge \sum_{n=1}^{\infty}|a_n|^2.$$

Proof. First we have to show that T is well defined. Let

$$T_{k,m}f = \sum_{n=k}^{m} \langle f, f_n\rangle\, f_n.$$

Observe that

$$\|T_{k,m}f\|^4 = |\langle T_{k,m}f, T_{k,m}f\rangle|^2 = \left|\sum_{n=1}^{\infty} \langle f, f_n\rangle \langle f_n, T_{k,m}f\rangle\right|^2$$
$$\le \sum_{n=k}^{m}|\langle f, f_n\rangle|^2 \sum_{n=k}^{m}|\langle f_n, T_{k,m}f\rangle|^2 \le B^2\|f\|^2\|T_{k,m}f\|^2.$$

Thus we conclude that

$$\|T_{k,m}f\| \le B\|f\|,$$

and

$$\|T_{k,m}f\|^2 \le B\sum_{n=k}^{m}|\langle f, f_n\rangle|^2$$

From this we conclude that $T_{1,m}f$ converges as $m \to \infty$ to an element Tf where T is a linear operator satisfying

$$\|T\| \le B.$$

Next we observe that

$$\langle Tf, f \rangle = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \ge A\|f\|^2.$$

From this we first conclude that $\|Tf\| \ge A\|f\|$ which implies that the range of T is closed. If this range is not H there is a nonzero vector $h \in H$ orthogonal to it, but this is impossible because $\langle Th, h \rangle \ge A\|h\|^2 > 0$.

Next we show that T is self-adjoint. Let f and $g \in H$ be arbitrary. Then

$$\begin{aligned} \langle Tf, g \rangle = \sum_{n=1}^{\infty} \langle f, f_n \rangle \langle f_n, g \rangle = \sum_{n=1}^{\infty} \langle f, f_n \rangle \overline{\langle f_n, g \rangle} \\ = \left\langle f, \sum_{n=1}^{\infty} \langle f_n, g \rangle f_n \right\rangle = \langle f, Tg \rangle . \end{aligned}$$

By the definition of T we have

$$f = T(T^{-1}f) = \sum_{n=1} \left\langle T^{-1}f, f_n \right\rangle f_n = \sum_{n=1} \left\langle f, T^{-1}f_n, f \right\rangle_n .$$

$\square$

Theorem 23. *Let H be a separable Hilbert space and let $(f_n)_{n=1}^{\infty}$ be a frame in H. Let $g_n = T^{-1}f_n$ where T is the operator $Tf = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$. Then either $(f_n)_{n=1}^{\infty}$ is a Riesz basis for H (with $\langle f_n, g_m \rangle = 0$ if $n \ne m$ and 1 if $n = m$) or there is a number $k \ge 1$ such that $(f_n)_{\substack{n=1 \\ n \ne k}}^{\infty}$ is a frame.*

Proof. If for all m and $n \ge 1$ we have

$$\langle f_n, g_m \rangle = \begin{cases} 0, & \text{if } n \ne m, \\ 1, & \text{if } n = m, \end{cases}$$

then $(f_n)_{n=1}^{\infty}$ is a Riesz basis by Theorem 20.(v).

Suppose that for some $k \ge 1$ either $\langle f_k, g_k \rangle \ne 1$ or $\langle f_k, g_m \rangle \ne 0$ for some $m \ne k$. Since $(f_n)_{n=1}^{\infty}$ is a frame we have write

$$f_k = \sum_{n=1}^{\infty} \langle f_k, g_n \rangle f_n.$$

If now $\langle f_k, g_k\rangle = 1$ then we have

$$0 = \sum_{\substack{n=1\\n\neq k}}^{\infty} \langle f_k, g_n\rangle\, f_n$$

On the other hand we have

$$0 = \sum_{n=1}^{\infty} 0\, f_n,$$

and by Theorem 22 we must therefore have

$$\langle f_k, g_n\rangle = 0, \quad n \neq k.$$

Thus we may assume that $a_k \stackrel{\text{def}}{=} \langle f_k, g_k\rangle \neq 1$. Then we have

$$f_k = \frac{1}{1-a_k} \sum_{\substack{n=1\\n\neq k}}^{\infty} \langle f_k, g_n\rangle\, f_n,$$

and in particular

$$|\langle f, f_k\rangle|^2 = \frac{1}{|1-a_k|^2} \left| \sum_{\substack{n=1\\n\neq k}}^{\infty} \overline{\langle f_k, g_n\rangle}\, \langle f, f_n\rangle \right| \\ \leq \frac{1}{|1-a_k|^2} \sum_{\substack{n=1\\n\neq k}}^{\infty} |\langle f_k, g_n\rangle|^2 \sum_{\substack{n=1\\n\neq k}}^{\infty} |\langle f, f_n\rangle|^2.$$

Thus we conclude that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq C\|f\|^2,$$

where $C = 1 + \frac{1}{|1-a_k|^2} \sum_{\substack{n=1\\n\neq k}}^{\infty} |\langle f_k, g_n\rangle|^2$. It follows that

$$\frac{A}{C}\|f\|^2 \leq \sum_{\substack{n=1\\n\neq k}}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2,$$

and we conclude that $(f_n)_{\substack{n=1\\n\neq k}}^{\infty}$ is a frame. This completes the proof. □

3. A frame of wavelets or ridgelets

Let $\alpha > 1$, for example $\alpha = 2$ and let $Q_d = [-\frac{1}{2}, \frac{1}{2}]^d$. Here we shall show that on gets a frame for the space $L^2(Q_d)$ in the form $\psi_{\mathbf{u},\alpha^j,\beta k\alpha^j}$ where

$$\psi_{\mathbf{u},a,b}(\underline{\mathbf{x}}) = \frac{1}{\sqrt{a}}\psi\left(\frac{\mathbf{u}\cdot\underline{\mathbf{x}} - b}{a}\right).$$

Here A_j is a set of vectors approximately uniformly distributed on the unit sphere, such that the number of vectors in A_j is of the order $\alpha^{-j(d-1)}$ when $j \to -\infty$. It is not difficult to show that one can get a similar frame for $L^2(K)$ where K is any bounded measurable set.

Theorem 24. *Assume that $d \geq 1$, $\alpha > 1$, $\beta > 0$, and that $\psi \in L^1(\mathbb{R})$ is such that for some $\delta > 0$*

$$\sup_{\omega\neq 0} \frac{|\hat{\psi}(\omega)|(1+|\omega|^{\frac{d+3}{2}+2\delta})}{|\omega|^{\frac{d-1}{2}+\delta}} < \infty, \tag{62}$$

and

$$\inf_{1\leq\omega\leq\alpha} \sum_{j=-\infty}^{j_0} \left(|\hat{\psi}(\alpha^j\omega)|^2 + |\hat{\psi}(-\alpha^j\omega)|^2\right) > 0. \tag{63}$$

Let j_* be such that $\alpha^{-j_*+1}d < \frac{1}{4} - \frac{1}{\pi}\arcsin(2^{\frac{1}{2}} - 2^{(\frac{1}{d}-\frac{1}{2})})$ and let $j_1 = j_0 + j_*$. Define the sets A_j as follows: $A_j = \cup_{p=j}^{j_1} B_p$ where

$$B_j = \left\{ \frac{1}{|\mathbf{v}|}\mathbf{v} \;\middle|\; \mathbf{v} \in \mathbb{Z}^d, \quad |\mathbf{v}|_\infty = \lceil \alpha^{-j+j_*} \rceil, \right. \\ \left. \min_{\mathbf{u}\in A_{j+1}} \left|\mathbf{u} - \tfrac{1}{|\mathbf{v}|}\mathbf{v}\right|_\infty \geq \frac{1}{d}\alpha^{j-j_1} \right\}. \tag{64}$$

If β is sufficiently small, then the functions

$$\{\psi_{\mathbf{u},\alpha^j,\beta k\alpha^j}\}_{(j\leq j_0,\mathbf{u}\in A_j,k\in\mathbb{Z})},$$

form a frame for $L^2(Q_d)$.

The set A_j as defined above is unnecessary large, and it is not difficult to construct much smaller sets A_j without loosing the frame-property.

We have the following result, which is a multidimensional version of the so-called Kadec's $\frac{1}{4}$-Theorem.

Theorem 25. *Let $d \geq 1$. Suppose that for each $\mathbf{k} \in \mathbb{Z}^d$ we have $|\omega_{\mathbf{k}} - \mathbf{k}|_\infty \leq L$. If $L \leq \frac{1}{2}$, then*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\hat{f}(\omega_{\mathbf{k}})|^2 \leq (2 - \cos(\pi L) + \sin(\pi L))^{2d}\|f\|^2_{L^2(Q_d)}, \quad f \in L^2(Q_d), \tag{65}$$

and if $L < \frac{1}{4} - \frac{1}{\pi}\arcsin(2^{\frac{1}{2}} - 2^{(\frac{1}{d}-\frac{1}{2})})$, *(so that is* $(2-\cos(\pi L)+\sin(\pi L))^d < 2$*) then*

(66)
$$\left(2 - (2-\cos(\pi L)+\sin(\pi L))^d\right)^2 \|f\|^2_{L^2(Q_d)} \le \sum_{\mathbf{k}\in\mathbb{Z}^d} |\hat{f}(\omega_{\mathbf{k}})|^2, \quad f \in L^2(Q_d).$$

In particular

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \sup_{|\mathbf{z}-\mathbf{k}|_\infty \le \frac{1}{2}} |\hat{f}(\mathbf{z})|^2 \le 8^d \|f\|^2_{L^2(Q_d)}, \quad f \in L^2(Q_d) \tag{67}$$

and when $L < \frac{1}{4} - \frac{1}{\pi}\arcsin(2^{\frac{1}{2}} - 2^{(\frac{1}{d}-\frac{1}{2})})$,

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \inf_{|\mathbf{z}-\mathbf{k}|_\infty \le L} |\hat{f}(\mathbf{z})|^2 \ge \left(2 - (2-\cos(\pi L)+\sin(\pi L))^d\right)^2 \|f\|^2_{L^2(Q_d)} \tag{68}$$

for every $f \in L^2(Q_d)$.

Proof of Theorem 25. Let

$$\lambda_d = (2-\cos(\pi L)+\sin(\pi L))^d - 1.$$

If we can prove that

$$\left\| \sum_{\mathbf{k}\in Z^d} c_{\mathbf{k}} \left(\mathrm{e}^{\mathrm{i}2\pi\mathbf{k}\cdot\underline{\mathbf{s}}} - \mathrm{e}^{\mathrm{i}2\pi\omega_{\mathbf{k}}\cdot\underline{\mathbf{s}}}\right) \right\|_{L^2(Q_d)} \le \lambda_d \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{k}}|^2}, \tag{69}$$

then it follows from Plancherel's theorem that if $T(\mathrm{e}^{\mathrm{i}2\pi\mathbf{k}\cdot\underline{\mathbf{s}}}) = \mathrm{e}^{\mathrm{i}2\pi\omega_{\mathbf{k}}\cdot\underline{\mathbf{s}}}$, then $\|T\| \le \lambda_d + 1$. If furthermore $\lambda_d < 1$, then it follows from [**8**, Thm. 1.10], that the sequence $(\mathrm{e}^{\mathrm{i}2\pi\omega_{\mathbf{k}}\cdot\underline{\mathbf{s}}})_{\mathbf{k}\in\mathbb{Z}^d}$ is a Riesz basis in $L^2(Q)$ and $\|T^{-1}\| \le \frac{1}{1-\lambda_d}$. From these inequalities the first two claims follow, so it remains to prove (68).

We use induction and note that if $d = 1$, then the claim is Kadec's $\frac{1}{4}$-Theorem, see [**8**][Thm 1.14]. If $d > 1$, $\mathbf{k} \in Z^d$, and $\mathbf{s} \in Q_d$, then we write $\mathbf{k} = (\mathbf{m}, n)$ where $\mathbf{m} \in Z^{d-1}$ and $n \in \mathbb{Z}$, $\mathbf{s} = (\mathbf{t}, u)$ where $\mathbf{t} \in Q_{d-1}$ and

$u \in [-\frac{1}{2}, \frac{1}{2}]$, and $\omega_{\mathbf{k}} = (\mu_{\mathbf{m}}, \eta_n)$. With this notation we have

$$
\begin{aligned}
(70)\quad & \left\| \sum_{\mathbf{k}\in Z^d} c_{\mathbf{k}} \left(e^{i2\pi \mathbf{k}\cdot \underline{\mathbf{s}}} - e^{i2\pi \omega_{\mathbf{k}}\cdot \underline{\mathbf{s}}} \right) \right\|_{L^2(Q_d)} \\
&\le \left\| \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} \sum_{n\in\mathbb{Z}} c_{\mathbf{m},n} e^{i2\pi \mathbf{m}\cdot \underline{\mathbf{t}}} \left(e^{i2\pi n \underline{u}} - e^{i2\pi \eta_n \underline{u}} \right) \right\|_{L^2(Q_d)} \\
&\quad + \left\| \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} \sum_{n\in\mathbb{Z}} c_{\mathbf{m},n} \left(e^{i2\pi \mathbf{m}\cdot \underline{\mathbf{t}}} - e^{i2\pi \mu_{\mathbf{m}}\cdot \underline{\mathbf{t}}} \right) e^{i2\pi n \underline{u}} \right\|_{L^2(Q_d)} \\
&\quad + \left\| \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} \sum_{n\in\mathbb{Z}} c_{\mathbf{m},n} \left(e^{i2\pi \mathbf{m}\cdot \underline{\mathbf{t}}} - e^{i2\pi \mu_{\mathbf{m}}\cdot \underline{\mathbf{t}}} \right) \left(e^{i2\pi n \underline{u}} - e^{i2\pi \eta_n \underline{u}} \right) \right\|_{L^2(Q_d)} .
\end{aligned}
$$

Now we have, since we know the claim holds when $d = 1$,

$$
\begin{aligned}
& \left\| \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} \sum_{n\in\mathbb{Z}} c_{\mathbf{m},n} e^{i2\pi \mathbf{m}\cdot \underline{\mathbf{t}}} \left(e^{i2\pi n \underline{u}} - e^{i2\pi \eta_n \underline{u}} \right) \right\|_{L^2(Q_d)}^2 \\
&\qquad \le \lambda_1^2 \int_{Q_{d-1}} \sum_{n\in\mathbb{Z}} \left| \sum_{\mathbf{m}\in Z^{d-1}} c_{\mathbf{m},n} e^{i2\pi \mathbf{m}\cdot \mathbf{t}} \right|^2 d\mathbf{t} \\
&\qquad = \lambda_1^2 \sum_{n\in\mathbb{Z}} \int_{Q_{d-1}} \left| \sum_{\mathbf{m}\in Z^{d-1}} c_{\mathbf{m},n} e^{i2\pi \mathbf{m}\cdot \mathbf{t}} \right|^2 d\mathbf{t} \\
&\qquad\qquad = \lambda_1^2 \sum_{n\in\mathbb{Z}} \sum_{\mathbf{m}\in Z^{d-1}} |c_{\mathbf{m},n}|^2 = \lambda_1^2 \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{k}}|^2.
\end{aligned}
$$

In the same way we get

$$
\left\| \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} \sum_{n\in\mathbb{Z}} c_{\mathbf{m},n} \left(e^{i2\pi \mathbf{m}\cdot \underline{\mathbf{t}}} - e^{i2\pi \mu_{\mathbf{m}}\cdot \underline{\mathbf{t}}} \right) e^{i2\pi n \underline{u}} \right\|_{L^2(Q_d)}^2 \le \lambda_{d-1}^2 \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{k}}|^2,
$$

and

$$
\begin{aligned}
& \left\| \sum_{\mathbf{m}\in\mathbb{Z}^{d-1}} \sum_{n\in\mathbb{Z}} c_{\mathbf{m},n} \left(e^{i2\pi \mathbf{m}\cdot \underline{\mathbf{t}}} - e^{i2\pi \mu_{\mathbf{m}}\cdot \underline{\mathbf{t}}} \right) \left(e^{i2\pi n \underline{u}} - e^{i2\pi \eta_n \underline{u}} \right) \right\|_{L^2(Q_d)}^2 \\
&\qquad \le \lambda_1^2 \lambda_{d-1}^2 \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{k}}|^2.
\end{aligned}
$$

Combining these inequalities with (69) we get our claim by an easy calculation.

□

Proof of Theorem 24. Let

$$\varphi_j(\underline{t}) = \alpha^{-\frac{j}{2}}\overline{\psi(-\alpha^{-j}\underline{t})}.$$

It follows that

$$\widehat{\varphi_j}(\underline{\omega}) = \alpha^{\frac{j}{2}}\overline{\hat{\psi}(\underline{\omega})}. \tag{71}$$

It is easy to check that

$$\left\langle f, \psi_{\mathbf{u},\alpha^j,\beta k\alpha^j}\right\rangle = \big(P_{\mathbf{u}} * \varphi_j\big)(k\beta\alpha^j).$$

We use the notation

$$F_j(\underline{t}) = \left|\big(P_{\mathbf{u}} * \varphi_j\big)(\underline{t})\right|^2.$$

Now if $t \in \mathbb{R}$ and $\tau > 0$ we have

$$\begin{aligned}\left|F_j(t) - \frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} F_j(s)\,\mathrm{d}s\right| &= \left|\frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\int_s^t F_j'(r)\,\mathrm{d}r\,\mathrm{d}s\right| \\ &= \left|\frac{1}{\tau}\int_{t-\frac{\tau}{2}}^{t} F_j'(r)\int_{t-\frac{\tau}{2}}^{t}\mathrm{d}s\,\mathrm{d}r - \frac{1}{\tau}\int_t^{t+\frac{\tau}{2}} F_j'(r)\int_t^{t+\frac{\tau}{2}}\mathrm{d}s\,\mathrm{d}r\right| \\ &\le \frac{1}{2}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}|F_j'(r)|\,\mathrm{d}r.\end{aligned}$$

Now we choose $t = k\beta\alpha^j$ for some $k \in \mathbb{Z}$ and $\tau = \beta\alpha^j$ so that

$$\left|F_j(k\beta\alpha^j) - \frac{1}{\beta\alpha^j}\int_{(k-\frac{1}{2})\beta\alpha^j}^{(k-\frac{1}{2})\beta\alpha^j} F_j(s)\,\mathrm{d}s\right| \le \frac{1}{2}\int_{(k-\frac{1}{2})\beta\alpha^j}^{(k-\frac{1}{2})\beta\alpha^j}|F_j'(s)|\,\mathrm{d}s.$$

Summing over $k \in \mathbb{Z}$ gives

$$\begin{aligned}&\left|\sum_{k\in\mathbb{Z}} F_j(k\beta\alpha^j) - \frac{1}{\beta\alpha^j}\int_{\mathbb{R}} F_j(s)\,\mathrm{d}s\right| \\ &\qquad \le \sum_{k\in\mathbb{Z}}\left|F_j(k\beta\alpha^j) - \frac{1}{\beta\alpha^j}\int_{(k-\frac{1}{2})\beta\alpha^j}^{(k-\frac{1}{2})\beta\alpha^j} F_j(s)\,\mathrm{d}s\right| \le \frac{1}{2}\int_{\mathbb{R}}|F_j'(s)|\,\mathrm{d}s.\end{aligned} \tag{72}$$

By Plancherel's theorem, (54), and (70) we have

$$\int_R F_j(s)\,\mathrm{d}s = \alpha^j\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega.$$

Now clearly

$$|F_j'(s)| \le 2|(P_{\mathbf{u}}f * \varphi_j)(s)||(P_{\mathbf{u}}f * \varphi_j')(s)|,$$

so that we get

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}|F_j'(s)|\,\mathrm{d}s &\le \left(\int_{\mathbb{R}}|(P_{\mathbf{u}}f*\varphi_j)(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|(P_{\mathbf{u}}f*\varphi_j)(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &= 2\pi\alpha^j\left(\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\omega|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}.\end{aligned}$$

Finally we sum over $\mathbf{u}$ and j with the result that

$$\begin{aligned}(73)\quad &\Bigg|\sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\sum_{k\in\mathbb{Z}}|\langle f,\psi_{\mathbf{u},\alpha^j,\beta k\alpha^j}\rangle|^2\\ &\qquad -\frac{1}{\beta}\sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\Bigg|\\ &\le \sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\left|\sum_{k\in\mathbb{Z}}|\langle f,\psi_{\mathbf{u},\alpha^j,\beta k\alpha^j}\rangle|^2-\frac{1}{\beta}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right|\\ &\le \sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}2\pi\alpha^j\left(\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\\ &\qquad\times\left(\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\omega|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\\ &\le 2\pi\left(\sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\\ &\qquad\times\left(\sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\alpha^j\omega|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}.\end{aligned}$$

Next we have to show that there is a positive constant c such that

$$(74)\qquad \sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega \le c\|f\|^2_{L^2(Q_d)},$$

$$(75)\qquad \sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\alpha^j\omega|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega \le c\|f\|^2_{L^2(Q_d)},$$

$$(76)\qquad \sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega \ge \frac{1}{c}\|f\|^2_{L^2(Q_d)}.$$

If this is the case, then it follows from (72) that

$$\left(\frac{1}{c\beta} - c\right) \|f\|^2_{L^2(Q_d)} \leq \sum_{j=-\infty}^{j_1} \sum_{\mathbf{u}\in A_j} \sum_{k\in\mathbb{Z}} |\langle f, \psi_{\mathbf{u},\alpha^j,\beta k\alpha^j}\rangle|^2$$
$$\leq \left(2\pi c + \tfrac{c}{\beta})\right) \|f\|^2_{L^2(Q_d)},$$

which completes the proof since we may choose $\beta < \frac{1}{c^2}$.

Let $\phi(\underline{\omega}) = |\hat{\psi}(\underline{\omega})|^2 + |\hat{\psi}(-\underline{\omega})|^2$. Since each set A_j is symmetric with respect to the mapping $\mathbf{u} \mapsto -\mathbf{u}$ we conclude that

$$\sum_{j=-\infty}^{j_1} \sum_{\mathbf{u}\in A_j} \int_{\mathbb{R}} |\hat{f}(\omega\mathbf{u})|^2 |\hat{\psi}(\alpha^j\omega)|^2 \,\mathrm{d}\omega = \sum_{j=-\infty}^{j_1} \sum_{\mathbf{u}\in A_j} \int_0^\infty |\hat{f}(\omega\mathbf{u})|^2 \phi(\alpha^j\omega) \,\mathrm{d}\omega.$$

By the definition of the sets A_j we get

$$\sum_{j=-\infty}^{j_1} \sum_{\mathbf{u}\in A_j} \int_0^\infty |\hat{f}(\omega\mathbf{u})|^2 \phi(\alpha^j\omega) \,\mathrm{d}\omega$$
$$= \sum_{j=-\infty}^{j_1} \sum_{p=j}^{j_1} \sum_{\mathbf{u}\in B_p} \sum_{k=-\infty}^{\infty} \int_{\alpha^k}^{\alpha^{k+1}} |\hat{f}(\omega\mathbf{u})|^2 \phi(\alpha^j\omega) \,\mathrm{d}\omega$$
$$= \sum_{k=-\infty}^{\infty} \sum_{p=-\infty}^{j_1} \sum_{\mathbf{u}\in B_p} \int_{\alpha^k}^{\alpha^{k+1}} |\hat{f}(\omega\mathbf{u})|^2 \sum_{j=-\infty}^{p} \phi(\alpha^j\omega) \,\mathrm{d}\omega.$$

Let $L = d\alpha^{-j_*+1}$ so that $L < \frac{1}{4} - \frac{1}{\pi}\arcsin(2^{\frac{1}{2}} - 2^{(\frac{1}{d}-\frac{1}{2})})$ and choose $k_0 = -j_*$ so that $\alpha^{k_0} < L$. Let c_3 be a positive constant such that

$$\sum_{j=-\infty}^{j_0} \phi(\alpha^j\omega) \geq c_3.$$

Since $k_0 + j_1 = j_0$ we have

$$\sum_{j=-\infty}^{j_1} \sum_{\mathbf{u}\in A_j} \int_{\mathbb{R}} |\hat{f}(\omega\mathbf{u})|^2 |\hat{\psi}(\alpha^j\omega)|^2 \,\mathrm{d}\omega \geq \sum_{k=k_0}^{\infty} \sum_{\mathbf{u}\in A_{j_0-k}} c_3 \int_{\alpha^k}^{\alpha^{k+1}} |\hat{f}(\omega\mathbf{u})|^2 \,\mathrm{d}\omega.$$

Now our construction of the sets A_j and our choice of k_0 guarantees that there is a positive constant c_4 so that for each $\mathbf{k} \in \mathbb{Z}^d$ there exists a $k \geq k_0$ and a vector $\mathbf{u} \in A_{j_0-k}$ such that the measure of the set $\{\, \omega \in [\alpha^k, \alpha^{k+1}] \mid$

$|\omega\mathbf{u}-\mathbf{k}|_\infty \le L\,\}$ is at least c_4. It follows that

$$\sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2|\hat{\psi}(\alpha^j\omega)|^2\,\mathrm{d}\omega \ge c_3c_4\sum_{\mathbf{k}\in\mathbb{Z}^d}\inf_{|\mathbf{z}-\mathbf{k}|_\infty\le L}|\hat{f}(\mathbf{z})|^2.$$

By (67) we get the desired lower bound (75).

In order to establish the upper bounds (73) and (74) we let $\phi(\underline{\omega}) = |\omega|\left(|\hat{\psi}(\underline{\omega})|^2+|\hat{\psi}(-\underline{\omega})|^2\right)$ and proceed as above to get

$$\begin{aligned}\sum_{j=-\infty}^{j_1}\sum_{\mathbf{u}\in A_j}&\int_{\mathbb{R}}|\hat{f}(\omega\mathbf{u})|^2\left(|\hat{\psi}(\alpha^j\omega)|^2+|\alpha^j\omega|^2|\hat{\psi}(\alpha^j\omega)|^2\right)\,\mathrm{d}\omega\\ &= \sum_{k=-\infty}^{\infty}\sum_{p=-\infty}^{j_1}\sum_{\mathbf{u}\in B_p}\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2\sum_{j=-\infty}^{p}\phi(\alpha^j\omega)\,\mathrm{d}\omega.\end{aligned}$$

It follows from our assumptions in (62) on ψ that $\sum_{j=-\infty}^{p}\phi(\alpha^j\omega)\le c_5<\infty$ for all $\omega\ge 0$. But we also have another constant c_6 such that

$$\sum_{j=-\infty}^{p}\phi(\alpha^j\omega)\le c_6\alpha^{(k+p)(d-1+2\delta)},\quad k+p\le 0,\quad \omega\le\alpha^{k+1}.$$

Since $\sup_{\mathbf{z}\in\mathbb{R}^d}|\hat{f}(\mathbf{z})|^2\le\|f\|^2_{L^2(Q_d)}$, we conclude that

$$\sum_{k=-\infty}^{-1}\sum_{p=-\infty}^{j_1}\sum_{\mathbf{u}\in B_p}\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2\sum_{j=-\infty}^{p}\phi(\alpha^j\omega)\,\mathrm{d}\omega\le c_7\|f\|^2_{L^2(Q_d)}.$$

On the other hand we have

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{p=-\infty}^{j_1}\sum_{\mathbf{u}\in B_p}&\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2\\ &\le\sum_{k=0}^{\infty}\sum_{p=-k+1}^{j_1}\sum_{\mathbf{u}\in B_p}\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2c_5\,\mathrm{d}\omega\\ &+\sum_{k=0}^{\infty}\sum_{p=-\infty}^{-k}\sum_{\mathbf{u}\in B_p}\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2c_6\alpha^{(k+p)(d-1+2\delta)}\,\mathrm{d}\omega\\ &= c_5\sum_{k=0}^{\infty}\sum_{\mathbf{u}\in B_{-k+1}}\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2\,\mathrm{d}\omega\\ &+c_6\sum_{k=0}^{\infty}\sum_{p=-\infty}-k\sum_{\mathbf{u}\in B_p}\int_{\alpha^k}^{\alpha^{k+1}}|\hat{f}(\omega\mathbf{u})|^2\alpha^{(k+p)(d-1+2\delta)}\,\mathrm{d}\omega.\end{aligned}$$

From the definition of the the sets A_j it follows that there is a constant c_8 such that

$$\sum_{\mathbf{u}\in B_{-k+1}} \int_{\alpha^k}^{\alpha^{k+1}} |\hat{f}(\omega\mathbf{u})|^2 \,\mathrm{d}\omega \le c_8 \sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ \alpha^k \le |\mathbf{k}| < \alpha^{k+1}}} \sup_{|\mathbf{z}-\mathbf{k}|_\infty \le \frac{1}{2}} |\hat{f}(\mathbf{z})|^2.$$

This takes care of the first term. Furthermore, we see that we can choose c_8 so that we also have

$$\sum_{\mathbf{u}\in B_p} \int_{\alpha^k}^{\alpha^{k+1}} |\hat{f}(\omega\mathbf{u})|^2 \,\mathrm{d}\omega \le c_8 \alpha^{-(p+k)(d-1)} \sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ \alpha^k \le |\mathbf{k}| < \alpha^{k+1}}} \sup_{|\mathbf{z}-\mathbf{k}|_\infty \le \frac{1}{2}} |\hat{f}(\mathbf{z})|^2.$$

Using this inequality we get the desired inequalities (73) and (74) and the proof is completed. $\square$