Kolmogorov's theorem on the representation of functions

1. Statement of the theorem

The purpose of this chapter is to prove a theorem due to Kolmogorov et al. which says that the continuous function of several variables can be given as compositions of continuous functions of one variable and addition. In other words, as long as one restricts onself to continuous functions, the only "genuine" function of two variables is $(x, y) \mapsto x + y$.

Theorem 12. *Let $d > 1$ and let K be a compact subset of $\mathbb{R}^d$. Then there exist d numbers $\lambda_j \in (0, 1)$, $j = 1, 2, \ldots, d$ and $2d + 1$ strictly increasing functions $\phi_k \in \mathcal{C}(\mathbb{R})$, $k = 1, 2, \ldots, 2d+1$ such that for each $f \in \mathcal{C}(\mathbb{R}^d)$ there is a function $g \in \mathcal{C}(\mathbb{R})$ such that*

$$f(x_1, x_2, \ldots, x_d) = \sum_{k=1}^{2d+1} g\left(\sum_{j=1}^{d} \lambda_j \phi_k(x_j)\right), \quad (x_1, x_2, \ldots, x_d) \in K. \tag{4}$$

In fact we will prove much more, namely that the only restriction on the numbers λ_j is that they are rationally independent, and the functions ϕ_k, can (once the λ_j have been chosen) be taken from a countable intersection of open sets in the space of nondecreasing function.

For completeness let us recall the definition of rational independence.

Definition 13. *A set $\Lambda \subset \mathbb{R}$ is rationally independent if it follows from*

$$\sum_{j=1}^{n} r_j \lambda_j = 0,$$

where $\lambda_j \in \Lambda$, $\lambda_j \neq \lambda_k$, and $r_j \in \mathbb{Q}$ when $j,k = 1,2,\dots n$ that

$$r_1 = r_2 = \dots = r_n = 0.$$

2. Proof of Theorem 12

We let G denote the set on nondecreasing functions in $\mathcal{C}(\mathbb{R})$ and G^m is the product of m copies of G with the metric

$$d\big((\phi_1,\dots,\phi_m),(\psi_1,\dots,\psi_m)\big) = \max_{1\le j\le m} d(\phi_j,\psi_j).$$

First we state and prove an "approximative" version of the theorem.

Lemma 14. *Assume that $d \geq 1$, $K \subset \mathbb{R}^d$ is compact, $\lambda_1,\dots,\lambda_d \in (0,1)$ are rationally independent, $f \in \mathcal{C}(\mathbb{R}^d)$ and let $m = 2d+1$. Define $\Gamma(f)$ to be the set of all $(\phi_1,\dots,\phi_m) \in G^m$ for which there is a function $h \in \mathcal{C}(\mathbb{R})$ such that*

$$\|h\|_{\mathcal{B}^\infty(\mathbb{R})} \leq \frac{1}{m}\|f\|_{\mathcal{B}^\infty(K)}, \tag{5}$$

$$\Big\| f - \sum_{k=1}^{m} h(\Phi_k)\Big\|_{\mathcal{B}^\infty(K)} < (1-\tfrac{1}{2m})\|f\|_{\mathcal{B}^\infty(K)} \quad \text{if} \quad f \neq 0, \tag{6}$$

where

$$\Phi_k(x_1,\dots,x_d) = \sum_{j=1}^{d} \lambda_j \phi_k(x_j), \quad (x_1,\dots,x_d) \in \mathbb{R}^d. \tag{7}$$

Then the set $\Gamma(f)$ is open and dense in G^m.

This lemma is the difficult part of the proof.

Proof of Lemma 14. Since the inequality in (6) is strict, it follows that $\Gamma(f)$ is open. (If $f = 0$ then $\Gamma(f) = G^m$.) In order to show that $\Gamma(f)$ is dense we let $(\psi_1,\dots,\psi_m) \in G^m$ and $\epsilon > 0$ be arbitrary, and we shall show that there is an element $(\phi_1,\phi_2,\dots,\phi_m) \in \Gamma(f)$ such that $d(\phi_k,\psi_k) < \epsilon$ for $k = 1,\dots,m$.

Let $a < b$ be some numbers such that

$$K \subset [a,b]^d.$$

Next we choose a number $\delta \in (0,1)$ such that

$$(8)\quad |f(x_1,\dots,x_d) - f(y_1,\dots,y_d| \le \frac{1}{2m}\|f\|_{\mathcal{B}^\infty(K)}, \qquad (x_1,\dots,x_d),(y_1,\dots,y_d) \in K, \quad \max_{1\le j\le k}|x_j - y_j| \le m\delta,$$

and

$$(9)\qquad \max_{1\le k\le m}|\psi_k(x) - \psi_k(y)| < \frac{\epsilon}{2}, \quad x,y\in[a,b], \quad |x-y|\le m\delta.$$

Define the intervals $I_k(i)$ by

$$I_k(i) = \Big[(k-1)\delta + mi\delta, (k-1)\delta + mi\delta + (m-1)\delta\Big), \quad k = 1,2,\dots,m, \quad i\in\mathbb{Z}.$$

Thus we see that each interval of this form has length $(m-1)\delta$ and there is a gap of size δ between $I_k(i)$ and $I_k(i+1)$. If $x\in\mathbb{R}$ there is thus for each k at most one i such that $x\in I_k(i)$. Moreover

$$(10)\qquad x\in\cup_{i\in\mathbb{Z}}I_k(i) \Leftrightarrow \left(\left\lfloor\frac{x}{\delta}\right\rfloor - k + 1\right) \mod m \in \{0,1,\dots,m-2\},$$

and there are $m-1$ numbers k for which this is the case and one for which it is not.

Furthermore we define the cubes

$$C_k(\mathbf{i}) = I_k(i_1)\times I_k(i_2)\times I_k(i_d), \quad \mathbf{i} = (i_1,i_2,\dots i_d).$$

If $\mathbf{x} = (x_1,\dots,x_d)\in\mathbb{R}^d$ then clearly there is at most one d-tuple of indices $\mathbf{i} = (i_1,\dots,i_d)$ such that $\mathbf{x}\in C_k(\mathbf{i})$. By (10) we see that $\mathbf{x}\in\cup_{\mathbf{i}\in\mathbb{Z}^d}C_k(\mathbf{i})$ if and only if $(\lfloor\frac{x_j}{\delta}\rfloor - k+1) \mod m \in\{0,1,\dots,m-2\}$, for each $j=1,\dots,d$. Since $\lfloor\frac{x_i}{\delta}\rfloor \mod m$ can only obtain d different values and $m = d+1+d$ there must be $d+1$ values of k so that $m-1$ is not among the numbers $(\lfloor\frac{x_j}{\delta}\rfloor - k+1) \mod m$ for all $j=1,\dots,d$. In other words, if $\mathbf{x}\in K$ and

$$(11)\qquad Q_{\mathbf{x}} = \{\,k \mid 1\le k\le m,\ \mathbf{x}\in\cup_{\mathbf{i}\in\mathbb{Z}^d}C_k(\mathbf{i})\,\} \quad\text{and}\quad Q'_{\mathbf{x}} = \{1,\dots,m\}\setminus Q_{\mathbf{x}},$$

then

$$(12)\qquad \#Q_{\mathbf{x}} \ge d+1 \quad\text{and}\quad \#Q'_{\mathbf{x}}\le d.$$

Let

$$(13)\qquad S_k = \{\,\mathbf{i} = (i_1,\dots,i_d) \mid C_k(\mathbf{i})\cap F \ne\emptyset\,\}, \quad k=1,\dots,m.$$

If $k\in\{1,m\}$ then we define

$$(14)\quad U_k = \{\,i\in\mathbb{Z} \mid \text{there is some } (i_1,\dots,i_d)\in S_k \text{ with } i = i_j, 1\le j\le d\,\}.$$

Let p_k be the number of elements in U_k. Assume that $n \in 1, \dots, m$ and that for $1 \le k \le n-1$ we have chosen numbers $y_{k,i}$, $i \in U_k$ such that all the numbers

$$a_{k,\mathbf{i}} = \sum_{j=1}^{d} \lambda_j y_{k,i_j}, \quad \mathbf{i} \in S_k, \quad k \in \{1, \dots, n-1\} \tag{15}$$

are different, such that $y_{k,i} \le y_{k,i'}$ if i, $i' \in U_k$ and $i \le i'$, and such that

$$|\psi_k((k-1)\delta + mi\delta) - y_{k,i}| < \frac{\epsilon}{2}. \tag{16}$$

Now we claim that we can choose $y_{n,i}$, $i \in U_n$ so that these statements hold for all $k \le n$. Let p be the number of elements in U_n. First we observe that each equation

$$\mathbf{a}_{n,\mathbf{i}} = \mathbf{a}_{k,\mathbf{i}'}, \quad k < n, \quad \mathbf{i}' \in S_k,$$

is a hyperplane in $\mathbb{R}^p$. Next we observe that each equation

$$\mathbf{a}_{n,\mathbf{i}} = \mathbf{a}_{n,\mathbf{i}'}$$

is either the equation of a hyperplane in $\mathbb{R}^p$ or an identity. We claim that it cannot be an identity unless $\mathbf{i} = \mathbf{i}'$. Suppose that $\mathbf{i} \ne \mathbf{i}'$ and let $j_* \in \{1, \dots, d\}$ be an index such that $i_{j_*} \ne i'_{j_*}$. Let $V = \{\, j \mid i_j = i_{j_*} \,\}$ and $V' = \{\, j \mid i'_j = i_{j_*} \,\}$. If we have an identity, then we must have

$$\sum_{j \in V} \lambda_j - \sum_{j' \in V'} \lambda_{j'} = 0.$$

But this is impossible since the numbers λ_j are rationally independent and $j_* \in V \setminus V'$. Thus we conclude that all the numbers $\mathbf{a}_{k,\mathbf{i}}$ are different for $k = 1, \dots, n$ provided the points $(y_{n,i})_{i \in U_n}$ do not belong to the union of a finite number of hyperplanes in $\mathbb{R}^p$. But since the complement of a finite union of hyperplanes is dense, we can choose the numbers $y_{n,i}$ so that the remaining conditions are satisfied as well. Thus the induction argument works and we get the conclusion for $n = m$ as well.

Now for $k = 1, \dots, m$ we define ϕ_k to be $y_{k,i}$ on the interval $I_k(i)$ if $i \in U_k$ and extend it so that it is nondecreasing and $\sup_{t \in \mathbb{R}} |\phi_k(t) - \psi_k(t)| < \epsilon$. By (9) and (16) this is possible.

If now $k \in \{1, \dots, m\}$ and $\mathbf{i} \in S_k$ then we choose some point $\mathbf{x}_{k,\mathbf{i}} \in F \cap C_k(\mathbf{i})$ and define

$$h(a_k(\mathbf{i})) = \frac{1}{m} f(x_{k,\mathbf{i}}). \tag{17}$$

For all remaining points we define h by linear interpolation, so that it is clear that (5) holds.

By our definitions of ϕ_k and h and by the requirement (8) it follows that if $\mathbf{x} \in C_k(\mathbf{i}) \cap F$ then

$$\left| \frac{1}{m} f(\mathbf{x}) - h(\Phi_k(\mathbf{x})) \right| \leq \frac{1}{2m^2} \|f\|_{\mathcal{B}^\infty(K)}. \tag{18}$$

Suppose now that $\mathbf{x} \in K$ is arbitrary and recall the definition of $Q_\mathbf{x}$ and $Q'_\mathbf{x}$. We have by (8), (12), (17) and (18)

$$\begin{aligned} \left| f(\mathbf{x}) - \sum_{k=1}^{m} h(\Phi_k(\mathbf{x}) \right| &= \left| \sum_{k=1}^{m} \left(\frac{1}{m} f(\mathbf{x}) - \sum_{k=1}^{m} h(\Phi_k(\mathbf{x}) \right) \right| \\ &\leq \sum_{k \in Q_\mathbf{x}} \left| \frac{1}{m} f(\mathbf{x}) - h(\Phi_k(\mathbf{x})) \right| + \sum_{k \in Q'_\mathbf{x}} \left| \frac{1}{m} f(\mathbf{x}) - h(\Phi_k(\mathbf{x})) \right| \\ &\leq \frac{m}{2m^2} \|f\|_{\mathcal{B}^\infty(K)} + (m-d-1) \left(\frac{1}{m} \|f\|_{\mathcal{B}^\infty(K)} + \|h(\Phi_k\|_{\mathcal{B}^\infty(K)} \right) \\ &\leq \frac{m}{2m^2} \|f\|_{\mathcal{B}^\infty(K)} + \frac{2(m-d-1)}{m} \|f\|_{\mathcal{B}^\infty(K)} \leq \left(1 - \frac{1}{2m}\right) \|f\|_{\mathcal{B}^\infty(K)}. \end{aligned} \tag{19}$$

Because $\mathbf{x}$ was arbitrary, we get the desired claim. □

Proof of Theorem 12. We let $m = 2d+1$ and take $\lambda_1, \ldots, \lambda_d \in (0,1)$ to be rationally independent. Let F be a countable dense subset of $\mathcal{C}(K)$. (This can be taken as the restriction to K of a countably dense subset of $\mathcal{C}(\mathbb{R}^d)$ since every function in $\mathcal{C}(K)$ can be extended to a function in $\mathcal{C}(\mathbb{R}^d)$.) Furthermore, for $j \in \{1, \ldots, m\}$ and $r, q \in \mathbb{Q}$ with $r < q$ we let $\Psi(j, r, q)$ be the set of functions $(\phi_1, \ldots, \phi_m) \in G^m$ such that $\phi_j(q) > \phi_j(r)$. This set is open and dense in G^m.

Choose

$$(\phi_1, \ldots, \phi_m) \in \bigcap_{f_* \in F} \Gamma(f_*) \cap \bigcap_{\substack{1 \leq j \leq m \\ r, q \in \mathbb{Q}, r < q}} \Psi(j, r, q).$$

This is possible since the intersection of countably many dense open subsets of a complete metric space is nonempty by Baire's theorem.

Let $f \in \mathcal{C}(K)$ be arbitary and define $f_0 = f$. Now we claim that there are functions h_i, and f_i, $i = 1, 2, 3, \ldots$ such that

$$\begin{gathered} \|h_i\|_{\mathcal{B}^\infty(\mathbb{R})} \leq \left(1 - \tfrac{1}{4m}\right)^{i-1} \|f\|_{\mathcal{B}^\infty(K)} \\ \|f_i\|_{\mathcal{B}^\infty(K)} \leq \left(1 - \tfrac{1}{4m}\right)^{i} \|f\|_{\mathcal{B}^\infty(K)} \\ f_i = f_{i-1} - \sum_{k=1}^{m} h_i(\Phi_k), \end{gathered}$$

where we use the notation in (7). Suppose that we have constructed these functions for $i = 1, \ldots, n$. If $n = 0$ this is an empty statement.

If $f_n = 0$ we take $h_{n+1} = 0$ and otherwise we choose a function $f_* \in F$ such that

$$\|f_*\|_{\mathcal{B}^\infty(K)} \le \|f_n\|_{\mathcal{B}^\infty(K)} \quad \text{and} \quad \|f_* - f_n\|_{\mathcal{B}^\infty(K)} \le \frac{1}{4m}\|f_n\|_{\mathcal{B}^\infty(K)}. \tag{20}$$

Since $(\phi_1, \ldots, \phi_m) \in \Gamma(f_*)$ there is by Lemma 14 a function h_{n+1} such that

$$\|h_{n+1}\|_{\mathcal{B}^\infty(R)} \le \|f_*\|_{\mathcal{B}^\infty(K)} \tag{21}$$

and

$$\left\| f_* - \sum_{k=1}^{m} h(\Phi_k) \right\|_{\mathcal{B}^\infty(K)} \le (1 - \tfrac{1}{2m})\|f_*\|_{\mathcal{B}^\infty(K)}. \tag{22}$$

If we let

$$f_{n+1} = f_n - \sum_{k=1}^{m} h_{n+1}(\Phi_k),$$

then inequalities (20), (21), and (22) imply that

$$\|f_{n+1}\|_{\mathcal{B}^\infty(K)} \le \big(1 - \tfrac{1}{2m}\big)\|f_*\|_{\mathcal{B}^\infty(K)} + \tfrac{1}{4m}\|f_n\|_{\mathcal{B}^\infty(K)} \le (1 - \tfrac{1}{4m})\|f_n\|_{\mathcal{B}^\infty(K)},$$

and

$$\|h_{n+1}\|_{\mathcal{B}^\infty(\mathbb{R})} \le \|f_n\|_{\mathcal{B}^\infty(K)}.$$

It follows that an induction argument works.

As a consequence we have $\lim_{i\to\infty} f_i = 0$ and if we define the function g by

$$g = \sum_{i=1}^{\infty} h_i,$$

then the series converges. Thus we also get

$$0 = f_0 - \sum_{j=1}^{m} g(\Phi_k),$$

which is what we wanted to prove since $f = f_0$. □