1. Let $\mathbf{u} \in \mathbb{R}^d$ be such that $|\mathbf{u}| = 1$. Define the Radon-transform $P_\mathbf{u} f$ as follows:

$$(P_\mathbf{u} f)(t) = \int_{\mathbb{R}^{d-1}} f(t\mathbf{u} + U^\perp \mathbf{s})\,\mathrm{d}\mathbf{s},$$

where $U^\perp$ is a $d \times (d-1)$ matrix with columns that form an orthonormal basis for the subspace of vectors in $\mathbb{R}^d$ orthogonal to $\mathbf{u}$. Show that if $f \in L^1(\mathbb{R}^d)$, then $P_\mathbf{u} f \in L^1(\mathbb{R})$ and

$$\widehat{P_\mathbf{u} f}(\underline{\omega}) = \hat{f}(\underline{\omega}\mathbf{u}).$$

Solution: Let $Q = [\mathbf{u}, U^\perp]$, that is the first column of Q is $\mathbf{u}$ and the remaining columns form an orthonormal basis for the subspace of vectors orthogonal to $\mathbf{u}$. Thus Q is an orthogonal matrix. If we define the change of variables by $\mathbf{x} = Q\mathbf{t}$ and let $g(\underline{\mathbf{t}}) = f(Q\underline{\mathbf{t}})$, then we see that $g \in L^1(\mathbb{R}^d)$, and

$$(P_\mathbf{u} f)(t) = \int_\mathbb{R} \dots \int_\mathbb{R} g(t, t_2, \dots, t_d)\,\mathrm{d}t_2\,\mathrm{d}t_3 \,\dots\, \mathrm{d}t_d,$$

and it follows from Fubini's theorem that $P_\mathbf{u} f \in L^1(\mathbb{R})$.

Now

$$\begin{aligned}\widehat{P_\mathbf{u} f}(\omega) &= \int_\mathbb{R} \mathrm{e}^{-\mathrm{i}2\pi\omega t} \int_{\mathbb{R}^{d-1}} f(\mathbf{u}t + U^\perp \mathbf{s})\,\mathrm{d}\mathbf{s}\,\mathrm{d}t \\ &= \int_\mathbb{R} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-\mathrm{i}2\pi\omega\mathbf{u}\cdot(t\mathbf{u}+U^\perp\mathbf{s})} f(\mathbf{u}t + U^\perp \mathbf{s})\,\mathrm{d}t\,\mathrm{d}\mathbf{s} \\ &\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}2\pi\omega\mathbf{u}\cdot Q\mathbf{t}} f(Q\mathbf{t})\,\mathrm{d}\mathbf{t} = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}2\pi\omega\mathbf{u}\cdot\mathbf{t}} f(\mathbf{t})\,\mathrm{d}\mathbf{t} = \hat{f}(\omega\mathbf{u}).\end{aligned}$$

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2. Let H be a separable Hilbert space, let $(f_n)_{n=1}^\infty$ be a frame in H, and let $Tf = \sum_{n=1}^\infty \langle f, f_n\rangle f_n$. Assuming that it has been shown that T is a bounded selfadjoint operator with bounded inverse, show that $(T^{-1} f_n)_{n=1}^\infty$ is a frame as well.

Solution: Since T is self-adjoint, so is T^{-1} and hence

$$\sum_{n=1}^\infty |\langle f, T^{-1} f_n\rangle|^2 = \sum_{n=1}^\infty |\langle T^{-1} f, f_n\rangle|^2.$$

Since $(f_n)_{n=1}^\infty$ is a frame in H we have

$$A\|T^{-1}f\|^2 \le \sum_{n=1}^\infty |\langle T^{-1} f, f_n\rangle|^2 \le B\|T^{-1}f\|^2,$$

and because $\|T^{-1}f\| \ge \|T\|^{-1}\|f\|$ and $\|T^{-1}f\| \le \|T^{-1}\|\|f\|$ we get

$$\frac{A}{\|T\|^2}\|f\|^2 \le \sum_{n=1}^\infty |\langle f, T^{-1} f_n\rangle|^2 \le B\|T^{-1}\|^2\|f\|^2,$$

and it follows that $(T^{-1}f_n)_{n=1}^{\infty}$ is a frame as well.

Note that we then always have

$$f = \sum_{n=1}^{\infty} \left\langle f, T^{-1}f_n \right\rangle f_n = \sum_{n=1}^{\infty} \left\langle f, f_n \right\rangle T^{-1}f_n.$$

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3. Let H be a separable Hilbert space and let $(f_n)_{n=1}^{\infty}$ be a frame in H. Does it follow that

$$\sup_{n\geq 1}\|f_n\| < \infty?$$

Solution: Since $(f_n)_{n=1}^{\infty}$ is a frame in H it follows for each $j \geq 1$ that

$$\|f_j\|^4 = |\langle f_j, f_j\rangle|^2 \leq \sum_{n=1}^{\infty}|\langle f_j, f_n\rangle|^2 \leq B\|f_j\|^2,$$

and so

$$\|f_j\|^2 \leq B.$$

Thus we conclude that $\sup_{n\geq 1}\|f_n\| \leq \sqrt{B} < \infty$.
