HUT , Institute of mathematics Gripenberg
Mat-1.196 Mathematics of neural networks
Exercise 3
29.1–6.2.2002

1. Let $a < b$. Explain how one can choose a countable set Λ so that $\Lambda \subset (a,b)$ and Λ is rationally independent, that is, if $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$, $\lambda_j \neq \lambda_k$ when $j \neq k$, and if $\sum_{j=1}^n r_j \lambda_j = 0$ where $n \geq 1$ and r_j, $j = 1, 2, \ldots, n$ are rational, then $r_j = 0$ for all $j = 1, \ldots, n$.

Solution: Choose λ_1 to be some nonzero number in (a,b), for example $\lambda_1 = \frac{a+b}{2}$ provided $a \neq -b$. Suppose next that we have chosen the numbers $\lambda_1, \ldots, \lambda_n$ in (a,b) and that they are rationally independent. Consider now the set

$$A_n = \left\{ \sum_{j=1}^n q_j \lambda_j \;\middle|\; q_j \in \mathbb{Q}, \quad j = 1, \ldots, n \right\},$$

(where $\mathbb{Q}$ denotes the set of rational numbers). Since $\mathbb{Q}$ is countable, the set A_n must be countable too, and because the interval (a,b) is uncountable, there must be some number $\lambda_{n+1} \in (a,b) \setminus A_n$. If the numbers $\lambda_1, \ldots, \lambda_{n+1}$ are rationally dependent there are rational numbers r_j such that

$$\sum_{j=1}^{n+1} r_j \lambda_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} |r_j| > 0.$$

If $r_{n+1} = 0$ we conclude that the numbers $\lambda_1, \ldots, \lambda_n$ are rationally dependent, which is a contradiction, and if $r_{n+1} \neq 0$ we have

$$\lambda_{n+1} = \sum_{j=1}^n q_j \lambda_j, \quad \text{where} \quad q_j = -\frac{r_j}{r_{n+1}},$$

which is a contradiction, because it implies that $\lambda_{n+1} \in A_n$. Using induction we can thus construct the desired set Λ.

---

2. Assume that $\phi_j \in \mathcal{C}(\mathbb{R})$ are strictly increasing for $j = 1, 2, \ldots, m$. Show that there is $\psi \in \mathcal{C}(\mathbb{R})$ which is strictly increasing such that all functions $\phi_j \circ \psi$, $j = 1, \ldots, m$, are Lipschitz continuous with Lipschitz constant 1, that is $|\phi_j(\psi(t)) - \phi_j(\psi(s))| \leq |t - s|$.
Hint: Let ψ be the inverse of the of the arclength function of the curve $t \mapsto (\phi_1(t), \ldots, \phi_m(t))$.

Solution: Let $\sigma(0) = 0$,

$$\sigma(s) = \sup\left\{ \sum_{k=1}^n \sqrt{\sum_{j=1}^m (\phi_j(t_k) - \phi_j(t_{k-1}))^2} \;\middle|\; 0 = t_0 \leq t_1 \leq \ldots \leq t_n = s \right\}, \quad s > 0,$$

and

$$\sigma(s) = \sup\left\{ \sum_{k=1}^n \sqrt{\sum_{j=1}^m (\phi_j(t_k) - \phi_j(t_{k-1}))^2} \;\middle|\; s = t_0 \leq t_1 \leq \ldots \leq t_n = 0 \right\}, \quad s < 0.$$

It is clear that if $s_1 < s_2$ then

$$\sigma(s_2) - \sigma(s_1) = \sup\left\{ \sum_{k=1}^{n} \sqrt{\sum_{j=1}^{m} (\phi_j(t_k) - \phi_j(t_{k-1}))^2} \;\middle|\; s_1 = t_0 \le t_1 \le \ldots \le t_n = s_2 \right\}.$$

It is clear that σ is strictly increasing and it is continuous because

$$\sum_{k=1}^{n} \sqrt{\sum_{j=1}^{m} (\phi_j(t_k) - \phi_j(t_{k-1}))^2} \le \sum_{k=1}^{n} \sum_{j=1}^{m} (\phi_j(t_k) - \phi_j(t_{k-1})) = \sum_{j=1}^{m} (\phi_j(s_2) - \phi_j(s_1)),$$

if $s_1 = t_0 \le t_1 \le \ldots \le t_n = s_2$. Thus this function σ has a strictly increasing and continuous inverse function ψ.

From the expression for $\sigma(s_2) - \sigma(s_1)$ we get

$$\sigma(s_2) - \sigma(s_1) \ge \sup\left\{ \sum_{k=1}^{n} \sqrt{(\phi_j(t_k) - \phi_j(t_{k-1}))^2} \;\middle|\; s_1 = t_0 \le t_1 \le \ldots \le t_n = s_2 \right\}$$

$$= \sup\left\{ \sum_{k=1}^{n} (\phi_j(t_k) - \phi_j(t_{k-1})) \;\middle|\; s_1 = t_0 \le t_1 \le \ldots \le t_n = s_2 \right\} = \phi_j(s_2) - \phi_j(s_1),$$

and by taking $s_2 = \psi(t)$ and $s_1 = \psi(s)$ we get the desired conclusion.

---

3. Assuming that Kolmogorov's theorem holds, show that one can choose the functions ϕ_j to be Lipschitz continuous with Lipschitz constant 1.

Solution: Let ψ be a strictly increasing function such that $|\phi_j(\psi(t)) - \phi_j(\psi(s))| \le |t - s|$ and let σ be the inverse function of ψ. If now f is a continuous function on K where $K \subset R^d$ is compact, then we define $\psi(K) = \{\, (\psi(x_1), \ldots, \psi(x_d)) \mid (x_1, \ldots, x_d) \in K \,\}$ and f_σ by

$$f_\sigma(x_1, \ldots, x_d) = f(\sigma(x_1), \ldots, \sigma(x_d)), \quad (x_1, \ldots, x_d) \in \psi(K).$$

By Kolmogorovs theorem we can write

$$f_\sigma(x_1, \ldots, x_d) = \sum_{k=1}^{d} g\left(\sum_{j=1}^{d} \lambda_j \phi_k(x_j) \right),$$

and it follows that

$$f(x_1, \ldots, x_d) = \sum_{k=1}^{d} g\left(\sum_{j=1}^{d} \lambda_j \phi_k(\psi(x_j)) \right), \quad (x_1, \ldots, x_d) \in K.$$

By our choice of ψ the functions $\phi_j \circ \psi$ are Lipschitz continuous with constant 1.
