HUT , Institute of mathematics — Gripenberg
Mat-1.196 Mathematics of neural networks
Exercise 3
29.1–6.2.2002

1. Let $a < b$. Explain how one can choose a countable set Λ so that $\Lambda \subset (a,b)$ and Λ is rationally independent, that is, if $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$, $\lambda_j \neq \lambda_k$ when $j \neq k$, and if $\sum_{j=1}^{n} r_j \lambda_j = 0$ where $n \geq 1$ and r_j, $j = 1, 2, \dots, n$ are rational, then $r_j = 0$ for all $j = 1, \dots, n$.

2. Assume that $\phi_j \in \mathcal{C}(\mathbb{R})$ are strictly increasing for $j = 1, 2, \dots, m$. Show that there is $\psi \in \mathcal{C}(\mathbb{R})$ which is strictly increasing such that all functions $\phi_j \circ \psi$, $j = 1, \dots, m$, are Lipschitz continuous with Lipschitz constant 1, that is $|\phi_j(\psi(t)) - \phi_j(\psi(s))| \leq |t - s|$.
Hint: Let ψ be the inverse of the of the arclength function of the curve $t \mapsto (\phi_1(t), \dots, \phi_m(t))$.

3. Assuming that Kolmogorov's theorem holds, show that one can choose the functions ϕ_j to be Lipschitz continuous with Lipschitz constant 1.

Theorem 1. *Let $d > 1$ and $K \subset \mathbb{R}^d$ be compact. Then there exists d numbers $\lambda_j \in (0,1)$, $j = 1, 2, \dots, d$ and $2d+1$ strictly increasing functions ϕ_k, $k = 1, 2, \dots, 2d+1$ such that for each $f \in \mathcal{C}(K)$ there exists a function $g \in \mathcal{C}(\mathbb{R})$ such that*

$$f(x_1, x_2, \dots, x_d) = \sum_{k=1}^{2d+1} g\left(\sum_{j=1}^{d} \lambda_j \phi_k(x_j)\right), \quad (x_1, x_2, \dots, x_d) \in K. \tag{1}$$