Aalto University

MS-A0401 Foundations of discrete mathematics
Exercise 5
5–9.10.2015, week 41

Gripenberg, Potka, Pöllänen, Tuhkanen,Ääri

Return your solutions to the P-questions and answer the S-questions not later than 12.10.2015 at 16.
Remember to write your name, student number and group!

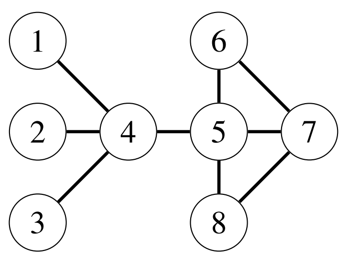
P1. The function $\alpha : \{0, 1, \dots, 13\} \to \{0, 1, 2 \dots, 13\}$, which is defined by the formula $\alpha(x) = \text{mod}\,(3 \cdot x, 14)$ is a bijection (because $\gcd(3, 14) = 1$).

(a) Determine the orbits of α, that is, the sets $\{\, \alpha^j(x) : j \geq 0 \,\}$ when $x \in \{0, 1, \dots, 13\}$ where $\alpha^j = \underbrace{\alpha \circ \alpha \circ \dots \circ \alpha}_{j}$.

(b) Write α as a product of cycles.

Hint: For example the cycle $\beta = (1\ 2\ 4)$ *is a bijection that satisfies the conditions* $\beta(1) = 2$, $\beta(2) = 4$, *and* $\beta(4) = 1$ *and* $\beta(x) = x$ *for all other* x *and its orbits are* $\{1, 2, 4\}$ *and the sets* $\{x\}$ *for all* $x \in \{0, 1, \dots, 13\} \setminus \{1, 2, 4\}$.

P2. Determine the group G that consist of all permutations f of the nodes in the graph below so that if there is an edge between nodes a and b (i.e., a and b are neighbours), then there is an edge between $f(a)$ and $f(b)$ as well, (i.e., they are neighbours as well).



Determine the cycle index $\zeta_{G,X} = \frac{1}{|G|} \sum_{f \in G} \zeta_{f,X}$ of this group where $\zeta_{f,X}(t_1, \dots, t_n) = t_1^{j_1} \cdot t_2^{j_2} \cdot \dots \cdot t_n^{j_n}$ when j_k is the number of orbits of length k of the permutation f.

Hint: If f *is a permuation of this kind then* $f(a)$ *has the same number of neighbours as* a.

Answer: $\zeta_{G,X}(t_1, t_2, t_3) = \frac{1}{12}\left(t_1^8 + 4t_1^6 t_2 + 3t_1^4 t_2^2 + 2t_1^5 t_3 + 2t_1^3 t_2 t_3\right)$

P3.

(a) Calculate, for example by listing all alternatives, in how many way you can give 3 red and 2 blue balloons to five people so that each one gets a red balloon or a blue balloon and the first two get balloons with same colour.

(b) Show that you get the answer to part (a) by calculating the coefficient of the term $r^3 b^2$ in the expression $(r^2 + b^2)(r + b)^3$.

(c) Calculate, by determining the coefficient of the term $r^5 b^4$ in a suitable expression (similar to the one in part (b)) in how many ways you can give 5 red and 4 blue balloons to nine people so that each one gets a red balloon or a blue balloon, the first two get balloons with same colour and also the following three get balloons with the same colour.

P4. How many different necklaces can one make using 3 white and 6 black pearls. When you decide which necklaces are the same you have to take into account both rotations and reflections, that is, the symmetry group is the dihedral group. Remember that the cycle index of the dihedral group D_n consisting of rotations an reflections of a regular polygon with n corners is

$$\zeta_{D_n,\mathbb{N}_n}(t_1, t_2, \ldots, t_n) = \frac{1}{2n}\left(\sum_{d\,|\,n} \varphi(d) t_d^{\frac{n}{d}}\right) + \begin{cases} \frac{1}{4}\left(t_2^{\frac{n}{2}} + t_1^2 t_2^{\frac{n}{2}-1}\right), & \text{if } n \text{ is even,} \\ \frac{1}{2} t_1 t_2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$

where $\varphi(d)$ is the number of integers j between 0 and $d-1$ so that $[j]_d$ is invertible in $\mathbb{Z}/d\mathbb{Z}$, that is, gcd $(j, d) = 1$.

Answer: 7

P5. The symmetric group S_3 consists of all permutations of the set $\{1, 2, 3\}$, that is $S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ where cycle notation is used.

(a) Show that $H = \{(1), (1\ 2)\}$ is a subgroup of S_3, that is that the products of the elements in H (i.e., the composite functions) belong to H and remember that (1) is the idenetity element.

(b) Determine the left cosets aH, $a \in S_3$ of H, that is determine the sets $\{a(1), a(1\ 2)\}$ for all $a \in S_3$.

(c) Determine with the aid of the results in part (b) elements so that $aH = bH$ and $cH = dH$ but $acH \neq bdH$.

Hint: Remember that when you calculate for example the product $(a\ b)(c\,d\,e)$ then you determine the composite function $f \circ g$ (i.e., "first g then f") where $f(a) = b$, $f(b) = a$ and $f(x) = x$ for all other x and $g(c) = d$, $g(d) = e$ and $g(e) = c$.

Note! *It follows from the results of part (c) that it is not possible to define an operation $\diamond$ on the left cosets so that $(xH) \diamond (yH) = xyH$. If you would calculate the right cosets Ha, $a \in S_3$ you would see that they are not the same as the left cosets and this subgroup H is not a so called normal subgroup!*