HELSINKI UNIVERSITY OF TECHNOLOGY
Institute of Mathematics
Mat-1.500 Martingale Theory, Spring 2003

Exercise IX
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1. Let Y_n be independent positive random variables. Let $\mathbb{E}(Y_n) = 1$. Define $X_n = \Pi_{i=1}^n Y_i$. Show that X_n is a martingale which converges a.s. to $X_\infty \in L^1$. Suppose specifically that
$$\mathbb{P}(Y_n = 1/2) = 1/2 = \mathbb{P}(Y_n = 3/2).$$
Show that $\mathbb{P}(X_\infty = 0) = 1$.

2. A martingale is said to be regular if $M_n = \mathbb{E}(X|\mathcal{F}_n)$ for for some $X \in L^1$. Is it true that a martingale is regular if and only if it converges in L^1?

3. Do the exercise on page 104.

4. Prove Lévy's downward theorem on page 136.

5. Let X be a uniformly distributed random variable on $[0, 1]$. Define
$$X_n = \frac{[2^n X]}{2^n},$$
where $[a]$ is the largest integer less than equal to a. Show that
$$\mathbb{E}(X_{n+1}|X_n) = X_n + \frac{1}{2^{n+2}}.$$