Mat-1.3608 Markov chains. For the exercises 1, 2 see the paper of Nummelin (a copy in the library) for definitions.

IV Exercise 14.2. 2008 Tikanmäki/Valkeila.

1. Check the details in the following identity:

$$\mathbb{E}_\nu \zeta_0(f) = \mathbb{E}_\nu \sum_{n=0}^{\tau-1} f(X_n) = \sum_{n=0}^{\infty} \int (\nu Q^n)(x) f(x) dx = \int \mu(x) f(x) dx.$$

2. Prove the formula

$$\mathbb{P}(\tau_i < \infty | X_0 = x) = \mathbb{P}(\tau < \infty | X_0 = x).$$

3. Let Y_n be a sequence of random variables with the property that

$$Y_n \to c \quad \text{a.s.} \quad \text{as} \quad n \to \infty$$

Let $N(n)$ be a sequence of random variables such that $N(n) \to \infty$ *a.s.* Check that $Y_{N(n)} \to c$ a.s, as $n \to \infty$. [The purpose of this exercise is to clarify some discussions from the lecture on Friday. Here $Y_n \to c$ a.s. means the following: $\mathbb{P}(\omega : Y_n(\omega) \to c) = 1$.]

4. Let π be the stationary distribution of a Markov chain with transition matrix P Let $A \subset S$, and define the truncation Q of P to A by $q_{ij} = p_{ij}$, if $s_i, s_j \in A$ and $i \neq j$, and when $i = j$ define

$$q_{ii} = p_{ii} + \sum_{k \in A^c} p_{ik}.$$

Show that if (P, π) is reversible, then also $(Q, \frac{\pi}{\pi(A)})$ is reversible.

5. Let X be a Markov chain with transition matrix P. Define $\tau_0 = 0$ and for $k \geq 0$ define $\tau_{k+1} = \min\{n \geq \tau_k + 1 : X_n \neq X_{\tau_k}\}$. Check that τ_k are stopping times and define $Y_n = X_{\tau_k}$ for all $n \geq 0$, where we put $X_{\tau_n} = \partial \notin S$, if $\tau_n = \infty$. Show that Y is a Markov chain and give its transition matrix.

6. Show that the Markov chain for the hard core model is irreducible and aperiodic [for more details, see Häggström, problem 7.1 and Chapter 7].