With the same step size, 0.1, the corresponding errors are given below.

t_n	Error	t_n	Error
0.1	0.050	1.1	15.93
0.2	0.337	1.2	13.25
0.3	1.123	1.3	10.18
0.4	2.642	1.4	7.28
0.5	5.328	1.5	4.87
0.6	8.909	1.6	3.06
0.7	12.731	1.7	1.81
0.8	15.899	1.8	1.01
0.9	17.617	1.9	0.54
1.0	17.570	2.0	0.27

Referring to Figure 8.3 we see that when $t \leq 1$, being above the correct solution is more serious than being below. By studying Figure 8.6 convince yourself that BE will place y_1 above $y(t_1)$ thus moving above the correct solution curve, but EM will be below it. Hence we expect BE to be less accurate than EM on this particular problem. These expectations are reflected in the errors listed for this example. ■

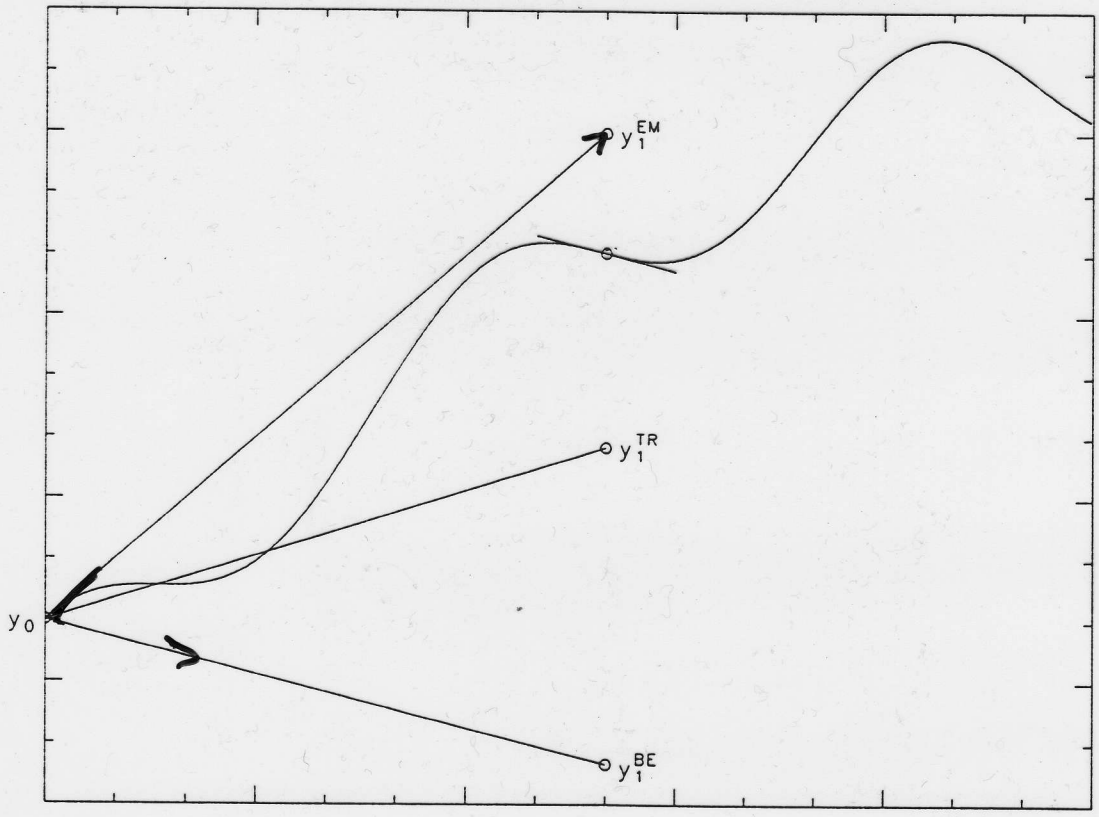
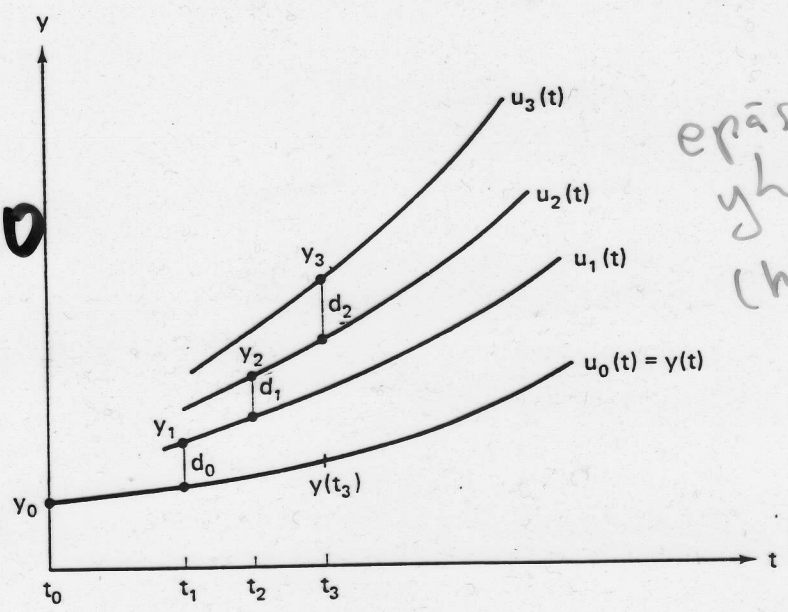


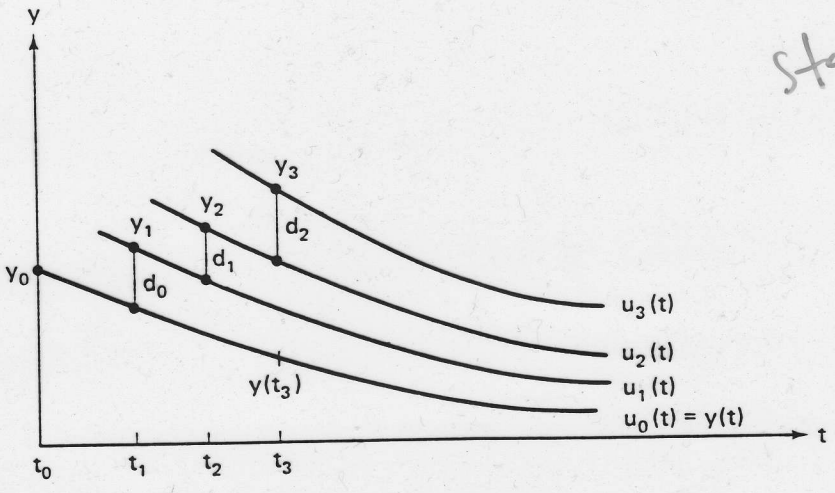
Figure 8.6 Graphical Interpretation of EM, BE and TR

Forsythe - Malcolm - Moler : Computer Methods for Math. Computations



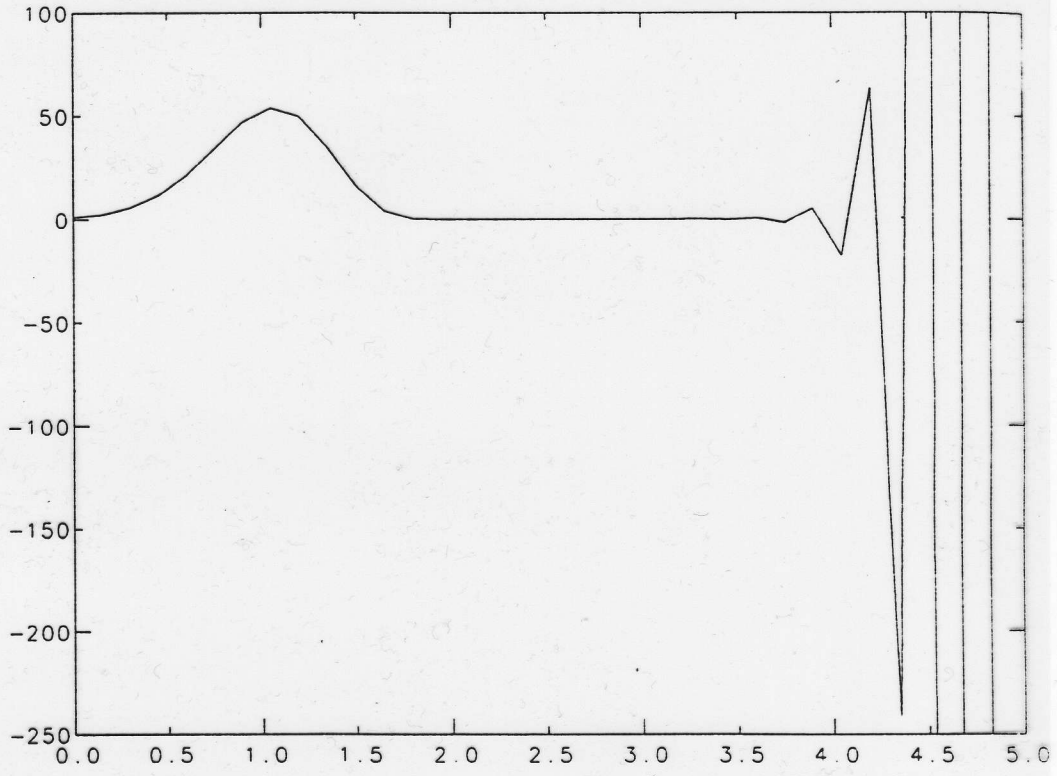
$$e_3 = y_3 - y(t_3) > d_0 + d_1 + d_2$$

Fig. 6.4. Local and global discretization errors for an unstable differential equation.



$$e_3 = y_3 - y(t_3) < d_0 + d_1 + d_2$$

Fig. 6.5. Local and global discretization errors for a stable differential equation.



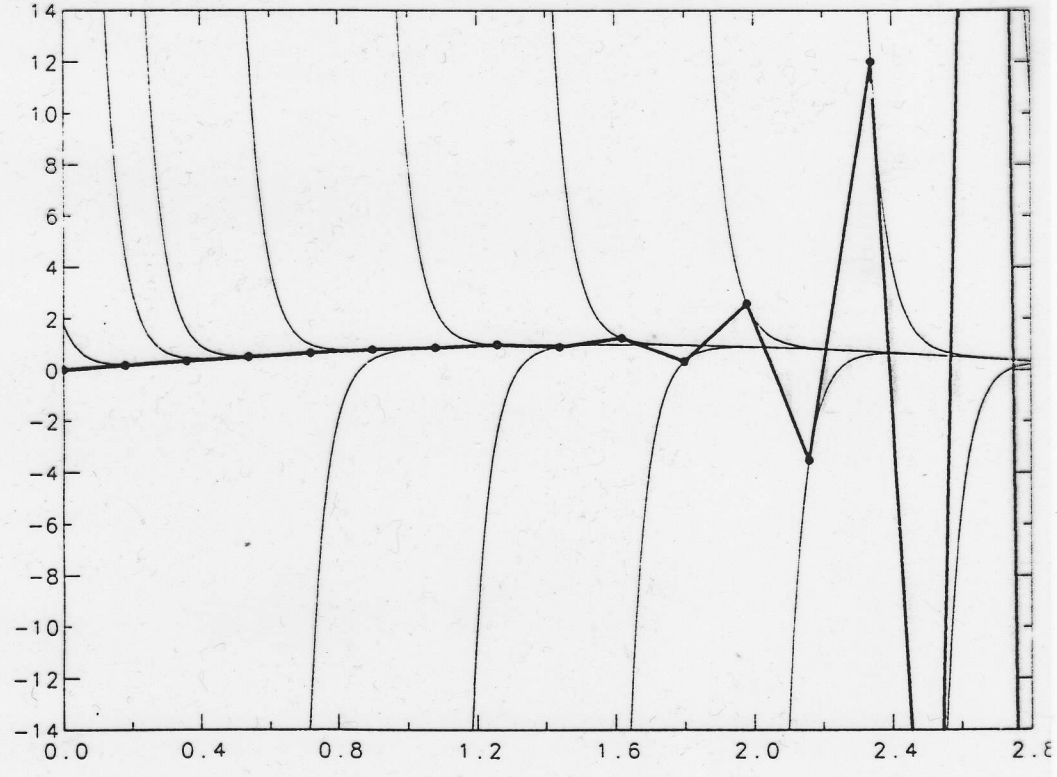


Figure 8.5 Euler's Method: (a) Becomes Unstable, (b) On A Stiff Problem

NUMERICAL RECIPES
The Art of Sci. Comp.

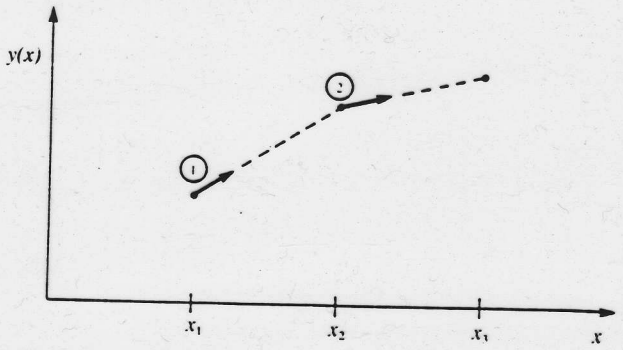


Figure 15.1.1. Euler's method. In this simplest (and least-accurate) method for integrating an ODE, the derivative at the starting point of each interval is extrapolated to find the next function value. The method has first-order accuracy.

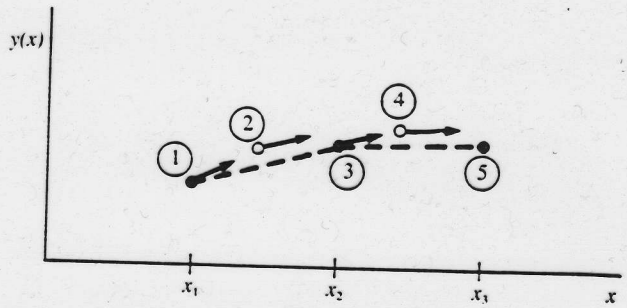


Figure 15.1.2. Midpoint method. Second order accuracy is obtained by using the initial derivative at each step to find a point halfway across the interval, then using the midpoint derivative across the full width of the interval. In the figure, filled dots represent final function values, while open dots represent function values that are discarded once their derivatives have been calculated and used.

Vrt. Atkinson
kuva s. 330

As indicated in the error term, this symmetrization cancels out the first order error term, making the method *second order*. [A method is conventionally called n^{th} order if its error term is $O(h^{n+1})$.] In fact, (15.1.2) is called the *second-order Runge-Kutta* or *midpoint* method.

We needn't stop there. There are many ways to evaluate the right-hand side $f'(x, y)$ which all agree to first order, but which have different coefficients of higher-order error terms. Adding up the right combination of these, we can eliminate the error terms order by order. That is the basic idea of the Runge-Kutta method. Abramowitz and Stegun, and Gear, give various specific formulas which derive from this basic idea. By far the most often used, and arguably even most useful, is the *fourth-order Runge-Kutta formula*, which has a certain sleekness of organization about it:

$$k_1 = hf'(x_n, y_n)$$
$$k_2 = hf'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = hf'(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$
$$k_4 = hf'(x_n + h, y_n + k_3)$$

RUNGE - KUTTA 4

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

missä

$$k_1 = h\,f(t_n, y_n)$$

$$k_2 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = h\,f\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_2\right)$$

$$k_4 = h\,f(t_{n+1}, y_n + k_3)$$

LKV $= O(h^5)$

Menetelmän kl = GKV $= O(h^4)$

(tai siis .. kertaluku = 4)