Differenssimenetelmä Laplace / Poisson -yhtälölle

TASOALUE

Laplace: $\nabla^2 u = u_{xx} + u_{yy} = 0$ (HY)

Poisson: $\nabla^2 u = u_{xx} + u_{yy} = f(x,y)$ (EHY)

$$u(x+h,y) = u(x,y) + h\,u_x(x,y) + \tfrac{1}{2}h^2 u_{xx}(x,y) + O(h^3)$$

$$u(x-h,y) = u(x,y) - h\,u_x(x,y) + \tfrac{1}{2}h^2 u_{xx}(x,y) + O(h^3)$$

$$\Rightarrow \quad u_x(x,y) \simeq \frac{1}{2h}\big(u(x+h,y) - u(x-h,y)\big)$$

$\vdots$

$$u_y(x,y) \simeq \frac{1}{2k}\big(u(x,y+k) - u(x,y-k)\big)$$

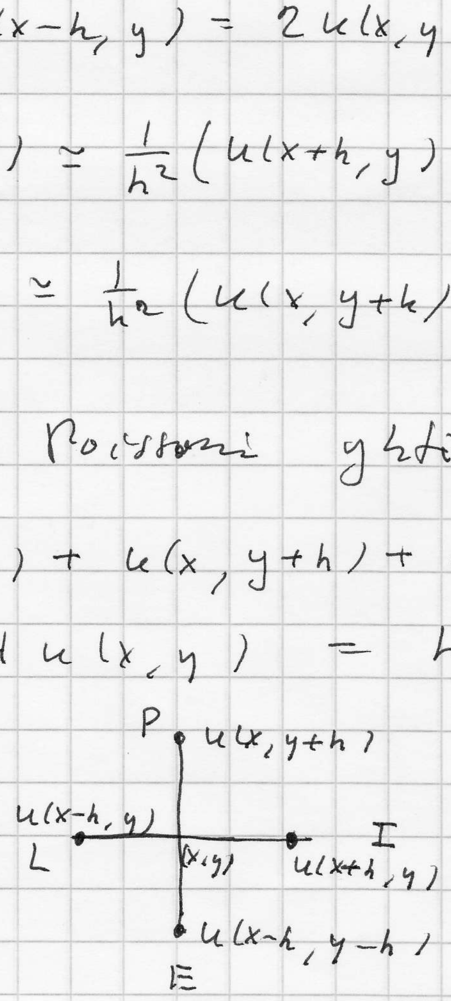
$$u(x+h,y) + u(x-h,y) = 2u(x,y) + h^2 u_{xx}(x,y)$$

$$\Rightarrow \quad u_{xx}(x,y) \simeq \frac{1}{h^2}\big(u(x+h,y) - 2u(x,y) + u(x-h,y)\big)$$

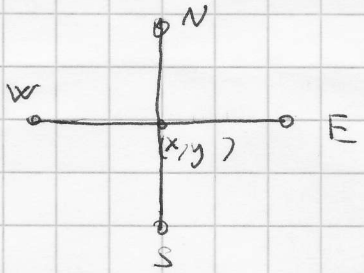
$$u_{yy}(x,y) \simeq \frac{1}{h^2}\big(u(x,y+k) - 2u(x,y) + u(x,y-k)\big)$$

Sijoitetaan Poissoni yhtälöön, $k = h$

$$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x-h,y-h) - 4u(x,y) = h^2 f(x,y)$$



$$u(I) + u(P) + u(L) + u(E) - 4u(x,y) = h^2 f(x,y)$$



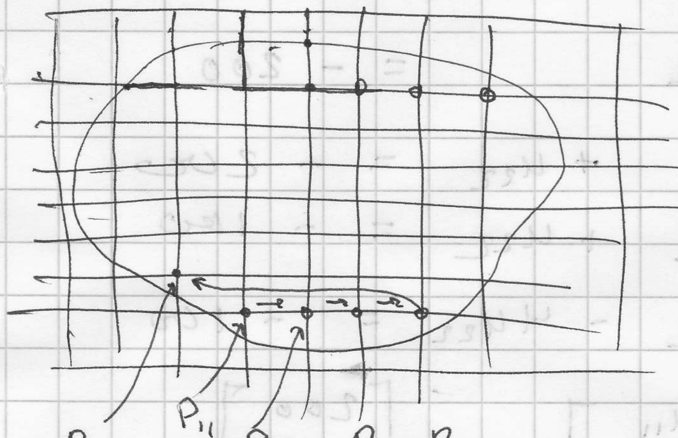
$$u(E) + u(N) + u(W) + u(S) - 4u(x,y) = h^2 f(x,y)$$

$$\left\{ \begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array} \right\} u = h^2 f(x,y)$$

5-point star, stencil, molecule

Dirichlet'n ongelma

Annettu tasoalueen reunalla jatkuva funktio f. Määritettävä funktio u, joka alueen sisällä on harmoninen, ts. $\Delta u = 0$ ja jonka reunalla yhtyy annettuun f:ään.



$p < 100$ pieni

$p > 100$: sparse

Iteratiivi

[Tällä kertaa jätetään väliin]

Gauss–Seidel

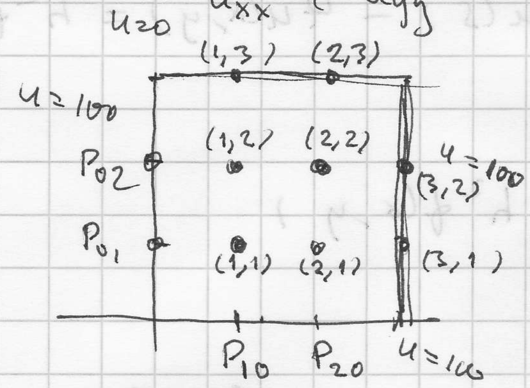
$\Longrightarrow$ Liebmann

[Ehkä ensin suora menetelmä, sitten sparse, ensin vähällä iteratiot]

$$u_t = c^2(u_{xx} + u_{yy})$$

Tasapainotilassa $u_t = 0$

$$u_{xx} + u_{yy} = 0$$



$$u_{ij} = \frac{1}{4}\left(u(E) + u(N) + u(W) + u(S)\right)$$

$$u_{11} = \frac{1}{4}\left(u_{21} + u_{12} + \underbrace{u_{01}}_{100} + \underbrace{u_{10}}_{100}\right)$$

$$u_{21} = \frac{1}{4}\left(\underbrace{u_{31}}_{100} + u_{22} + u_{11} + \underbrace{u_{20}}_{100}\right)$$

$$u_{12} = \frac{1}{4}\left(u_{22} + \underbrace{u_{13}}_{0} + \underbrace{u_{02}}_{100} + u_{11}\right)$$

$$u_{22} = \frac{1}{4}\left(\underbrace{u_{32}}_{100} + \underbrace{u_{23}}_{0} + u_{12} + u_{21}\right)$$

$\Rightarrow$

$$\begin{cases} -4u_{11} + u_{21} + u_{12} = -200 \\ u_{11} - 4u_{21} + u_{22} = -200 \\ u_{11} - 4u_{12} + u_{22} = -100 \\ u_{21} + u_{12} - 4u_{22} = -100 \end{cases}$$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = -\begin{bmatrix} 200 \\ 200 \\ 100 \\ 100 \end{bmatrix}$$

Laplacemuun1.html