Mat-1.1332 KP3-II Tentti 7.2.2009

Ratkaisut / Heikki Apiola

1) $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & \frac{3}{2} & 1 \\ 1 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}$ Muunnetaan Gaussin askelilla yläkolmiomuotoon U.

Alakolmiomatriisi $L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$ Luvut * ovat yllä käytetyt kertoimet vastalukuina. Sijoittuvat tukisarakkeen kohtaan, joka nollataan.

$\begin{bmatrix} 1 & -1 & -1 \\ -1 & \frac{3}{2} & 1 \\ 1 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}$ ① + ⊖1 + $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & * & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ ① + $\quad \underline{\underline{L}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$\searrow$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \underline{\underline{U}}$$

Nyt $A = LU$, missä L ja U yllä

Ratkaistava $A\vec{x} = \vec{b}$, $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$

Ts. $L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$.

Ratk ① $L\vec{y} = \vec{b}$ ja ② $U\vec{x} = \vec{y}$.

① $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix} \Rightarrow$

$\underline{y_1 = 1}$

$-\underbrace{y_1}_{1} + y_2 = -1$

$\Rightarrow \underline{y_2 = 0}$

$\underbrace{y_1}_{1} - \underbrace{y_2}_{0} + y_3 = \frac{1}{2}$

$\underline{y_3 = -\frac{1}{2}}$

2 $\begin{bmatrix} 1 & -1 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

$\Rightarrow \quad x_3 = 1, \quad x_2 = 0, \quad x_1 = 2; \quad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

---

2) a) $\begin{cases} 1x + 0y = 0 \\ 0x + 1y = 0 \\ 1x + 1y = 1 \end{cases}$ YLIMÄÄRÄYTYVÄ SYSTEEMI $\Leftrightarrow \quad A\vec{x} = \vec{b}$,

missä $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

PNS - ratkaisu: $A^T A \vec{x} = A^T \vec{b}$

$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

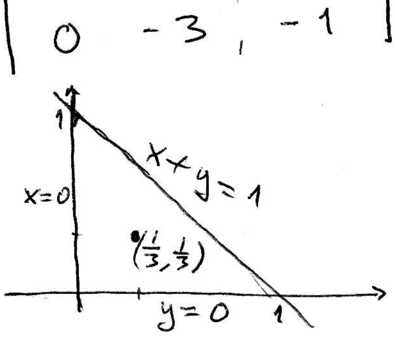
$A^T \vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 1 \end{array}\right] \begin{matrix} (-2) \\ + \end{matrix}$

$\sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & -1 \end{array}\right] \Rightarrow \underline{\underline{y = \frac{1}{3}}}$

$x + 2y = 1$

$\Rightarrow \underline{\underline{x = \frac{1}{3}}}$



2 b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \underbrace{\frac{1}{3!}A^3t^3 + \ldots}_{0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

3) $\vec{y}' = A\vec{y}$, $\vec{y}(0) = \vec{y}_0 = \begin{bmatrix} 1 \\ 2.4 \end{bmatrix}$

$A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$

Ominaisarvot : $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{vmatrix}$

$= (4-\lambda)(1-\lambda) - 10 = \lambda^2 - 5\lambda - 6$

$p(\lambda) = 0 \iff \lambda = \begin{cases} 6 \\ -1 \end{cases}$

Ominaisvektorit :

$\lambda_1 = 6$: $\begin{bmatrix} -2 & -5 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\iff 2x_1 + 5x_2 = 0 \iff x_1 = -\frac{5}{2} x_2$.

Val. $x_2 = 2 \Rightarrow x_1 = -5$.

Om. vekt. $\vec{v}_1 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

$\lambda_2 = -1$ $\begin{bmatrix} 5 & -5 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff x_1 - x_2 = 0$

Val. $x_2 = 1 \Rightarrow x_1 = 1$;

Om. vekt. $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

yleinen ratkaisu :

$\vec{y}(t) = c_1 e^{6t} \begin{bmatrix} -5 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Alkuehto: $\begin{bmatrix} 1 \\ 2.4 \end{bmatrix} = \vec{y}(0) = c_1 \begin{bmatrix} -5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\iff \begin{bmatrix} -5 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -5 & 1 & 1 \\ 2 & 1 & 2.4 \end{array}\right] \frac{2}{5} + \sim \begin{bmatrix} -5 & 1 & 1 \\ 0 & 1.4 & 2.8 \end{bmatrix}$$

$$1.4\, c_2 = 2.8 \implies c_2 = 2$$
$$-5 c_1 + \underbrace{c_2}_{2} = 1 \implies c_1 = \frac{1}{5}$$

$$\implies \vec{y}(t) = \frac{1}{5} e^{6t} \begin{bmatrix} -5 \\ 2 \end{bmatrix} + 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

---

4) $y'' + \cos y = 0$

Merk. $y_1 = y$, $y_2 = y' = y_1'$.

Tällöin yhtälö on yhtäpitävä systeemin

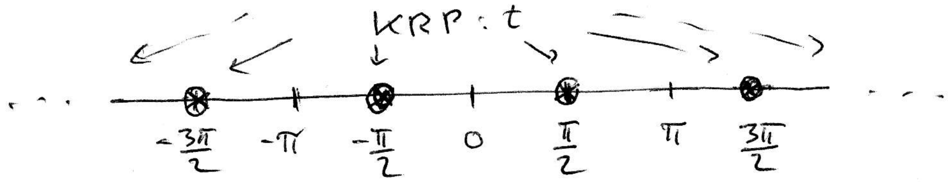
$$\boxed{\begin{cases} y_1' = y_2 \\ y_2' = -\cos y_1 \end{cases}}$$

kanssa.

Merk. $\begin{cases} \phi_1(y_1, y_2) = y_2 \\ \phi_2(y_1, y_2) = -\cos y_1 \end{cases}$

Kriittiset pisteet: $\begin{cases} \phi_1(y_1, y_2) = 0 \\ \phi_2(y_1, y_2) = 0 \end{cases}$

$$\iff \begin{cases} y_2 = 0 \\ \cos y_1 = 0 \end{cases} \iff y_1 = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$



Linearisointi

Jacobin matriisi :

$$J(y_1, y_2) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin y_1 & 0 \end{bmatrix}$$

KRP_1 : $(\frac{\pi}{2} + 2n\pi, 0)$, KRP_2 : $(-\frac{\pi}{2} + 2n\pi, 0)$

$$J_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Ominaisarvot

KRP_1 : J_1 : $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \underbrace{\lambda^2 - 1}_{=0 \Leftrightarrow \lambda = \pm 1.}$

Ominaisarvot reaaliset erimerkkiset $\Rightarrow$ kyseessä satulapiste, epästabiili. (linearisoidun systeemin)

KRP_2 : J_2 : $\det(J_2 - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$

$$= \underbrace{\lambda^2 + 1}_{=0} \Leftrightarrow \lambda = \pm i.$$

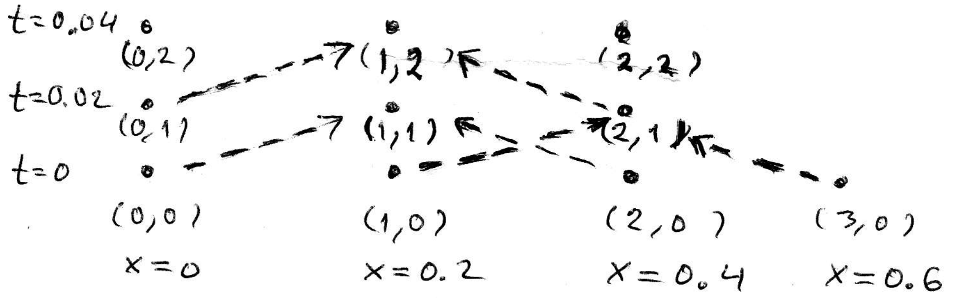
Ominaisarvot puhtaasti imaginaariset. KRP_2 :ssa linearisoitu systeemi on **keskus**, (heikosti) stabiili.

[Huom! Ominaisvektoreita ei tarvitse laskea, tyyppi ja stabiilisuus selviävät ominaisarvoista.]

5) Paikka - askel $h = 0.2$.

Aika - askel $k = rh^2 = \frac{1}{2} 0.04 = 0.02$

Päästäksemme aikatasolle $t = 0.04$, täytyy siis suorittaa 2 aika - askelta.



$u_{0,0} = 0$, $u_{1,0} = f(0.2) = 100 \sin(0.2\pi) = 58.78$

AIKATASO $t = 0$ (ALKUTILA)

$u_{2,0} = f(0.4) = 100 \sin(0.4\pi) = 95.11$

$u_{3,0} = f(0.6) = 100 \sin(0.6\pi) = 95.11$

AIKATASO $t = 0.02$, $j = 1$ $(t = jk, j = 1)$

$$u_{11} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 95.11) = 47.55$$

$$u_{21} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(58.78 + 95.11) = 76.94$$

AIKATASO $t = 0.04$, $j = 2$ $(t = jk, j = 2)$

$$u_{12} = \frac{1}{2}(u_{0,1} + u_{2,1}) = \frac{1}{2}(0 + 76.94)$$

$$= \boxed{38.47}$$