Mat-1.3345 Diff. yht. inversio-ongelmat MALLIT
HARJOITUS 6
22.11.2006

1. $u(x,t) = \sum_{j=1}^{\infty} a_j(t) \sin jx \quad (\Rightarrow u(0,t) = u(\pi,t) = 0)$

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{j=1}^{\infty} \left(a_j''(t) + j^2 a_j(t) \right) \sin jx$$
$$= f(x,t), \quad 0 < x < \pi$$

$$\Rightarrow a_j''(t) + j^2 a_j(t) = \frac{2}{\pi} \int_0^{\pi} f(x,t) \sin jx\, dx =: c_j(t)$$

Vakion variointi & alkuarvot $a_j(0) = a_j'(0) = 0$,
$\Rightarrow$ 1-käs. ratkaisu

$$a_j(t) = \frac{1}{j} \int_0^t \sin j(t-s)\, c_j(s)\, ds$$

Osasummille pätee, $u_N(x,t) = \sum_{j=1}^{N} a_j(t) \sin jx$

$$\| u - u_N \|_{H^2(0,\pi)} = \| \sum_{j>N} a_j(t) \sin(j\cdot) \|_{H^2(0,\pi)}$$
$$\leq \sum_{j>N} |a_j(t)|\, \| \sin(j\cdot) \|_{H^2(0,\pi)}$$

Voidaan laskea

$$\| \sin(j\cdot) \|^2_{H^2(0,\pi)} \leq C j^4 \pi$$

Kertoimille a_j saadaan

$$|a_j(t)| \leq \frac{1}{j} \int_0^t \frac{2}{\pi} \left| \int_0^{\pi} f(x,s) \sin jx\, dx \right| ds$$
$$= \frac{2}{\pi j} \int_0^t \left| \int_0^t \frac{\partial f}{\partial x}(x,s) \frac{\cos jx}{j} dx \right| ds$$
$$\overset{\text{lisää os. integrointia}}{\leq} \frac{2}{\pi j^4} \int_0^t \int_0^{\pi} \left| \frac{\partial^3 f}{\partial x^3}(x,s) \right| dx\, ds$$

$$\Rightarrow \| u - u_N \|_{H^2(0,\pi)} \leq \sum_{j>N} \frac{\sqrt{\pi(1+j^4)}}{\pi j^4} \left\| \frac{\partial^3 f}{\partial x^3} \right\|_{L^1((0,\pi)\times\mathbb{R}_+)} \to 0 \quad \square$$

2. $$Au := -\frac{d}{dx}\left(\frac{1}{(1+x)^2}\frac{d}{dx}u\right)$$

Nyt $mg^{-\frac{1}{2}} = \frac{1}{(1+x)^2}$ ja $m^{-1}g^{-\frac{1}{2}} = 1$,

joten $m(x) = \frac{1}{1+x}$ ja $g(x) = (1+x)^2$

Tehdään ensin koordinaatistonmuunnos

$$\tilde{x} = \tilde{X}(x) = \int_1^x (1+x')\,dx' = x + \frac{1}{2}x^2 - \frac{3}{2}$$

$$\Rightarrow X(\tilde{x}) = 2^{\frac{1}{2}}(\tilde{x}+2)^{\frac{1}{2}} - 1$$

$$\Rightarrow \tilde{m}(\tilde{x}) = 2^{-\frac{1}{2}}(\tilde{x}+2)^{-\frac{1}{2}}, \quad \tilde{g}(\tilde{x}) = 2(\tilde{x}+2)\cdot\left[2^{-\frac{1}{2}}(\tilde{x}+2)^{-\frac{1}{2}}\right]^2 = 1$$

Mittamuunnoksella $\kappa(\tilde{x}) = \tilde{m}^{\frac{1}{2}}(\tilde{x}) = 2^{-\frac{1}{4}}(\tilde{x}+2)^{-\frac{1}{4}}$ saadaan

$$\begin{aligned}
\tilde{A}_\kappa \tilde{u} &= \kappa \tilde{A}(\kappa^{-1}\tilde{u}) \\
&= 2^{-\frac{1}{4}}(\tilde{x}+2)^{-\frac{1}{4}} \cdot 2^{\frac{1}{2}}(\tilde{x}+2)^{\frac{1}{2}}\left(2^{-\frac{1}{2}}(\tilde{x}+2)^{-\frac{1}{2}}\left(2^{\frac{1}{4}}(\tilde{x}+2)^{\frac{1}{4}}\tilde{u}\right)'\right)' \\
&= (\tilde{x}+2)^{\frac{1}{4}}\left((\tilde{x}+2)^{-\frac{1}{2}}\left[\frac{1}{4}(\tilde{x}+2)^{-\frac{3}{4}}\tilde{u} + (\tilde{x}+2)^{\frac{1}{4}}\tilde{u}'\right]\right)' \\
&= (\tilde{x}+2)^{\frac{1}{4}}\left(\frac{1}{4}(\tilde{x}+2)^{-\frac{5}{4}}\tilde{u} + (\tilde{x}+2)^{-\frac{1}{4}}\tilde{u}'\right)' \\
&= (\tilde{x}+2)^{\frac{1}{4}}\left(\frac{1}{4}\cdot\left(-\frac{5}{4}\right)(\tilde{x}+\ )^{-\frac{9}{4}}\tilde{u} + (\tilde{x}+2)^{-\frac{1}{4}}\tilde{u}''\right) \\
&= \tilde{u}'' - \frac{5}{16}(\tilde{x}+2)^{-2}\tilde{u}
\end{aligned}$$

3. Osoita Lemma 1.9, kun aaltoyhtälössä termi $q(x)u$ korvataan termillä $p(x,t)\partial_x u + q(x,t)u$

Samaan tapaan kuin lemma 1.9:n todistuksessa:

$$\frac{dH_u}{dt} = \mathrm{Re}\left[\int_0^{\ell} \partial_t u\left(\overline{F} + (1+q)\bar{u} - p\,\overline{\partial_x u}\right)dx\right]$$

$$\leq \frac{1}{2}\|\partial_t u(\cdot,t)\|_{L^2}^2 + \frac{1}{2}\|F(\cdot,t)\|_{L^2}^2 + (C_q+1)\|u(\cdot,t)\|_{L^2}^2$$

$$+ C_p\|\partial_x u(\cdot,t)\|_{L^2}^2$$

$$\leq \frac{1}{2}\|F(\cdot,t)\|_{L^2}^2 + (C_p+C_q+2)H_u, \qquad (*)$$

missä $C_q = \max_{\substack{x\in[0,\ell]\\ t\in[0,T]}} q(x,t)$, $C_p = \max_{\substack{x\in[0,\ell]\\ t\in[0,T]}} p(x,t)$

$$(*) \Rightarrow \frac{d}{dt}\left(e^{-(C_p+C_q+2)t}H_u\right) \leq \frac{1}{2}e^{-(C_p+C_q+2)t}\|F(\cdot,t)\|^2$$

$$\Rightarrow H_u(t) \leq \frac{1}{2}\int_0^t e^{(C_p+C_q+2)(t-t')}\|F(\cdot,t')\|^2 dt'$$

$$\leq C(T)\|F\|_{L^2([0,\ell]\times[0,T])}^2$$

4. Osoita kaava (1.45) ratkaisemalla (1.48)-(1.49).

$$\begin{cases} \frac{d^2}{dt^2} u_k^f(t) = -\lambda_k u_k^f(t) + \varphi_k'(0) f(t) \\ u_k^f(0) = \frac{d}{dt} u_k^f(0) = 0 \end{cases}$$

Vakion variointi: selvitetään ensin homogeenisen yhtälön

$$\frac{d^2}{dt^2} u_k^f(t) + \lambda_k u_k^f(t) = 0$$

kaksi lineaarisesti riippumatonta ratkaisua $v_k(t)$ ja $w_k(t)$, jonka jälkeen yleinen ratkaisu on

$$u_k^f(t) = -\int_0^t \frac{1}{W} w_k(t') f(t') dt' \varphi_k'(0) v_k(t)$$
$$+ \int_0^t \frac{1}{W} v_k(t') f(t') dt' \varphi_k'(0) w_k(t),$$

missä W on Wronskin determinantti.

$\lambda_k > 0$: $v_k(t) = \sin\sqrt{\lambda_k} t$, $w_k(t) = \cos\sqrt{\lambda_k} t$

$$W = \begin{vmatrix} \sin\sqrt{\lambda_k} t & \cos\sqrt{\lambda_k} t \\ \sqrt{\lambda_k}\cos\sqrt{\lambda_k} t & -\sqrt{\lambda_k}\sin\sqrt{\lambda_k} t \end{vmatrix} = -\sqrt{\lambda_k}$$

Nyt $\sin(\sqrt{\lambda_k}(t-t')) = \sin(\sqrt{\lambda_k} t)\cos(\sqrt{\lambda_k} t') - \cos(\sqrt{\lambda_k} t)\sin(-\sqrt{\lambda_k} t')$ etc. joten

$$u_k^f(t) = \int_0^t \frac{1}{\sqrt{\lambda_k}} \cos(\sqrt{\lambda_k} t') f(t') dt' \varphi_k'(0) \sin(\sqrt{\lambda_k} t)$$
$$- \int_0^t \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t') f(t') dt' \varphi_k'(0) \cos(\sqrt{\lambda_k} t)$$
$$= \int_0^t \frac{\sin\sqrt{\lambda_k}(t-t')}{\sqrt{\lambda_k}} f(t') dt' \varphi_k'(0)$$

Muut tapaukset menevät vastaavasti:

$\lambda_k = 0$: $v_k(t) = t$, $w_k(t) = 1$

$\lambda_k < 0$: $v_k(t) = e^{\sqrt{\lambda_k} t}$, $w_k(t) = e^{-\sqrt{\lambda_k} t}$