Example: Target tracking with Kalman filter

Classical application of Kalman filters is target tracking, e.g., in GPS positioning.

$$r = r(t) \in \mathbb{R}^2$$

$$v = v(t) = \frac{dr}{dt}(t) \in \mathbb{R}^2.$$

Discrete time steps:

$$t_j = j\Delta t, \quad j = 0, 1, 2, \ldots$$

Discrete model:

$$r(t_{j+1}) = r(t_j) + \Delta t v(t_j) + \varepsilon_j,$$

where

$$\varepsilon_j = \text{modelling error.}$$

A priori model for the time evolution of the velocity:

$$v(t_{j+1}) = v(t_j) + \eta_j, \quad \eta_j \sim \mathcal{N}(0, \delta^2 I).$$

Stochastic model:

$$X_j = \begin{bmatrix} r(t_j) \\ v(t_j) \end{bmatrix} \in \mathbb{R}^4,$$

interpreted as random variable. Evolution model

$$\begin{aligned} X_{j+1} &= \begin{bmatrix} r(t_{j+1}) \\ v(t_{j+1}) \end{bmatrix} \\ &= \begin{bmatrix} r(t_j) + \Delta t v(t_j) + \varepsilon_j \\ v(t_j) + \eta_j \end{bmatrix} \\ &= \begin{bmatrix} I & \delta t I \\ 0 & I \end{bmatrix} \begin{bmatrix} r(t_j) \\ v(t_j) \end{bmatrix} + \begin{bmatrix} \varepsilon_j \\ \eta_j \end{bmatrix} \\ &= A X_j + E_j. \end{aligned}$$

Observation model

Assume that alternatingly, we observe either the first or the second component of the position:

$$y_j = \begin{cases} r_1(t_j) + w_j, & \text{if } j \text{ is even,} \\ r_2(t_j) + w_j, & \text{if } j \text{ is odd,} \end{cases}$$

where

$$w_j = \text{ measurement error}, \quad w_j \sim \mathcal{N}(0, \sigma^2).$$

Stochastic model:

$$Y_j = B_j X_j + W_j,$$

where

$$B_j = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad j \text{ even,}$$

$$B_j = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad j \text{ odd,}$$

Program

Defining the matrices for the model:

```
 dt = 0.1; % time step
 nt = 50;  % number of time steps

 % Propagation matrix A

A = [eye(2) dt*eye(2);zeros(2,2),eye(2)];

 % Observation matrix: observation is alternatingly the first
 % or the second component of the location

 BB = [1,0,0,0;0,1,0,0];
```

Defining the noise covariance matrices

```
% STD of the location (gamma) and velocity (eta)

gamma = 0.02;
eta = 0.1;

% STD of observation error

sigma = 0.= 30;

G = diag([gamma^2*ones(2,1);eta^2*ones(2,1)]);
Sigma = sigma^2;
```

Initialization:

```
% STD of the initial state and the initial state
delta = 1;
D = delta^2*eye(4);
x = zeros(4,1); t = 0;

TrueTrack = NaN*ones(2,nt);
TrueVelocity = NaN*ones(2,nt);
[r,v] = Target(t);
TrueTrack(:,1) = r;
TrueVelocity(:,1) = v;
```

```
EstTrack = NaN*ones(2,nt);
EstVelocity = NaN*ones(2,nt);
EstTrack(:,1) = x(1:2);
EstVelocity(:,1) = x(3:4);

PredTrack = NaN*ones(2,nt);
PredVelocity = NaN*ones(2,nt);
```

```
% Kalman filtering
for j = 2:nt
    % Prediction step
    xpred = A*x;
    Dpred = A*D*A' + G;

    PredTrack(:,j) = xpred(1:2);
    PredVelocity(:,j) = xpred(3:4);
```

```
    % Updating step
    t = t + dt;
    [r,v] = Target(t);
    TrueTrack(:,j) = r;
    TrueVelocity(:,j) = v;
    if mod(j,2) == 0
        B = BB(1,:);
        y = r(1) + sigma*randn;
    else
        B = BB(2,:);
        y = r(2) + sigma*randn;
    end
    K = Dpred*B'*inv(B*Dpred*B' + Sigma);
    x = xpred + K*(y - B*xpred);
    D = Dpred -K*B*Dpred;
    EstTrack(:,j) = x(1:2);
    EstVelocity(:,j) = x(3:4);
end
```

Results

Plotting the trajectories

$$t_j \mapsto r(t_j) \text{ (true trajectory)},$$

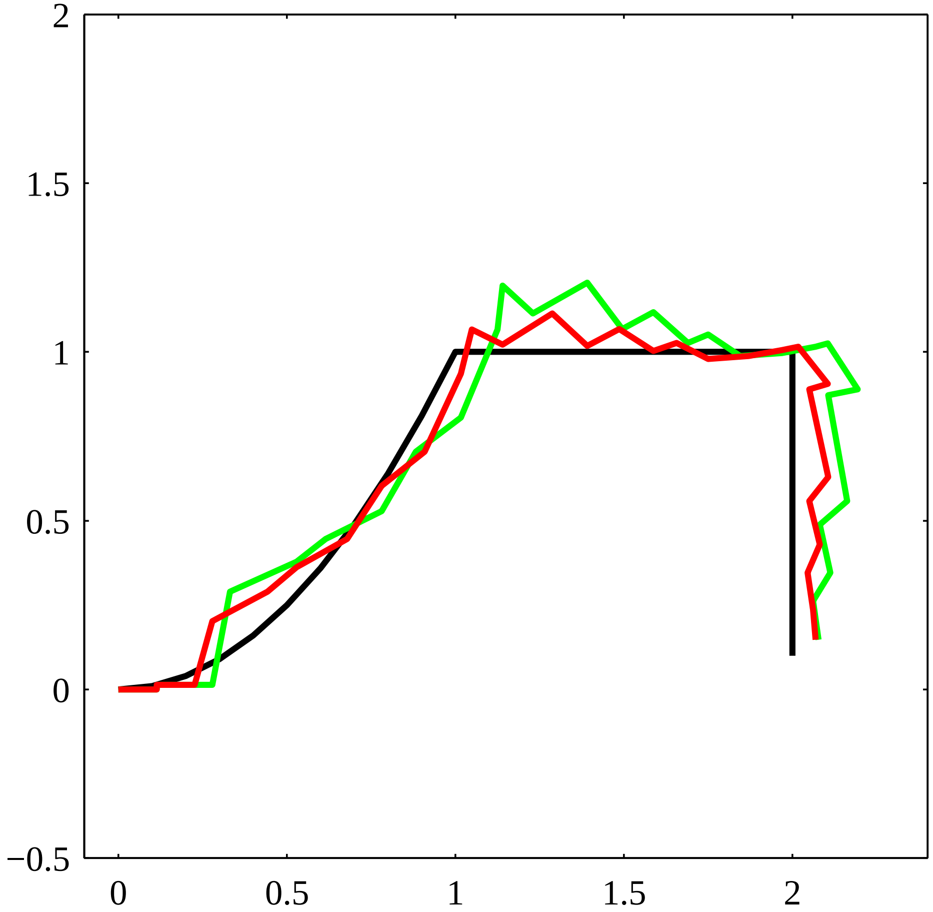
$$t_j \mapsto \widehat{r}(t_j) \text{ (predicted trajectory)},$$

and

$$t_j \mapsto \overline{r}(t_j) \text{ (estimated trajectory)}.$$

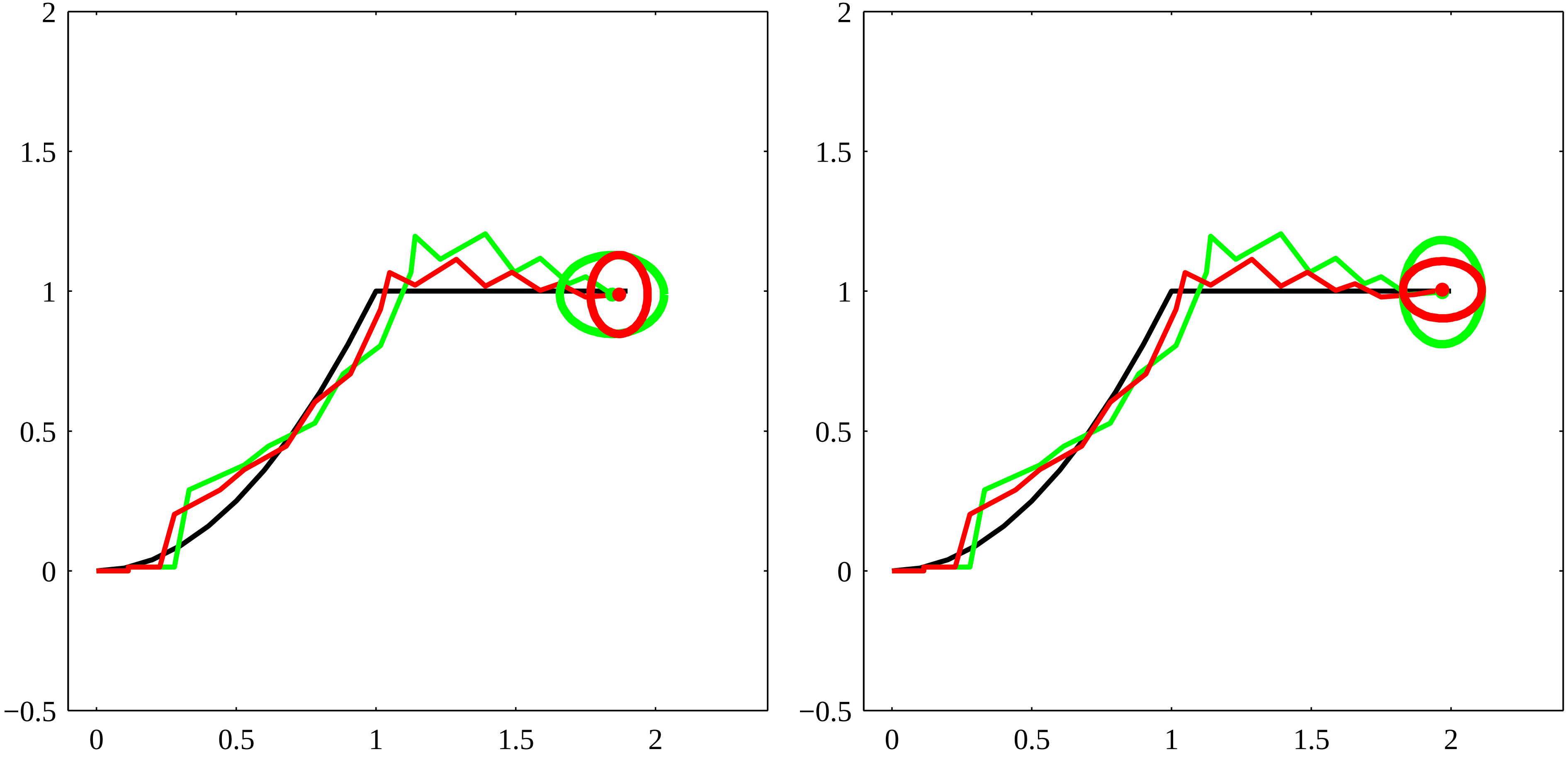
Snapshots of 95% belief ellipses at $j = 20, 21$.

Kalman tracking



Prediction (green) and estimation (red)

Consequtive snapshots of covariances



Prediction (green) and estimation (red)

Observations

- The estimation covariance is smaller than the prediction covariance:

$$D_{j+1} = \overline{D}_{j+1} - \underbrace{\overline{D}_{j+1} B^{\mathrm{T}} (B\overline{D}_{j+1} B^{\mathrm{T}} + \Sigma_{j+1})^{-1} B\overline{D}_{j+1}}_{\text{pos. definite}}.$$

- When the first component of the position is measured, the belief ellipse shrinks in horizontal direction, while when the second component is measured, the ellipse shrinks in the vertical direction.