Home Assignment 2

In this assignment, the goal is to study the coordinate transformations from the point of view of sampling.

1. First a simple one-dimensional example. Assume that $X$ is a real-valued random variable with the uniform probability density over the unit interval,

$$\pi(x) = \chi(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

   We define a new real valued random variable $Y$ by setting

$$Y = \frac{1}{X+1}.$$

   Write the probability density of $Y$ explicitly. Then demonstrate, that if you draw a sample $\{x_1, x_2, \ldots, x_n\}$ from the probability density of $X$ and map them to the points $x_k \mapsto y_k$, the $Y$–sample thus obtained approximates the density of $Y$. The conclusion is: when you are estimating distributions using sampling, you can ignore the computation of the Jacobians that you need to calculate when dealing with the distributions.

2. A two-dimensional example: consider the probability density discussed in the lecture notes (Chapter 9: "Sampling: the real thing"),

$$\pi(x_1, x_2) \propto \exp\left(\frac{1}{2\sigma^2}((x_1^2 + x_2^2)^{1/2} - 1)^2 - \frac{1}{2\delta^2}(x_2 - 1)^2\right),$$

   with $\sigma = 0.1$, $\delta = 1$.

   Making the change of variables to polar coordinates $(r, \theta)$, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, write the probability density in terms of $r$ and $\theta$. Then write a Gibbs sampler that samples this density in the $(r, \theta)$–space. Finally, map the sample points back to $(x_1, x_2)$–space.

   Estimate the mean of the density. Calculate also the autocovariances $\gamma_k$ of your sample.

3. Write a random walk Metropolis-Hastings algorithm in the $(r, \theta)$–space and map the sample points back to the $(x_1, x_2)$–space. Show a scatter plot of the result, as well as the sample

histories of both components. Try different step lengths, and record the acceptance rate.

In addition to showing the results, comment on the scatter plots and sample histories in exercises 2 and 3.