Mat-5.3741 Theory of Elasticity (5 cp) L

Spring 2007
Stenberg/Juntunen

Exercise 9

Problem 1

Eliminate the moments M_{ij} and the shear Q_i, $i, j = 1, 2$, in the equations

$$M_{11} = -D\left(\frac{\partial^2 w}{\partial x_1^2} + \nu\frac{\partial^2 w}{\partial x_2^2}\right) \qquad M_{22} = -D\left(\frac{\partial^2 w}{\partial x_2^2} + \nu\frac{\partial^2 w}{\partial x_1^2}\right)$$

$$M_{12} = -(1-\nu)D\frac{\partial^2 w}{\partial x_1 x_2}$$

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = Q_1 \qquad \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} = Q_2$$

$$-\left(\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2}\right) = f$$

What equation do you get for the deflection w?

Problem 2

Probably the simplest Kirchhoff plate finite element is Morley's nonconforming triangular element. The local space is $P_2(K)$, K = triangle, and the degrees of freedom are the values at the three vertices and the values of the normal derivative at the three midpoints of the edges. Show that this set of degrees of freedom is "unisolvent", i.e. determines the functions uniquely.

Problem 3 **(home exercise)**

Consider a circular domain $\Omega = \{x_1^2 + x_2^2 < R^2\}$ and a uniform load $f = 1/(\pi a^2)$ in the domain $x_1^2 + x_2^2 \leq a^2$. Solve the problem (e.g. with Maple/Mathematica) both for clamped and simply supported case. Print all variables. Study also the limit solution $a \to 0$.

Problem 4 **(home exercise to be handed on Tuesday April 24.)**

Consider a simply supported Kirchhoff plate in the domain $(0, 2a) \times (0, 2b)$. The load is uniform $f = 1/(2cd)$ in the region $(a-c, a+c) \times (b-d, b+d)$. Solve the problem by Fourier series and plot the deflection, the moments, and the shear force. Study also the limit solutions $(c, d) \to (0, 0)$ and $(c, d) \to (a, b)$.

Problem 1

$$M_{11} = -D\left(w^{(2,0)} + \nu w^{(0,2)}\right)$$

$$M_{22} = -D\left(w^{(0,2)} + \nu w^{(2,0)}\right)$$

$$M_{12} = -(1-\nu) D w^{(1,1)}$$

$$Q_1 = \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2}$$

$$= -D\left(w^{(3,0)} + \nu w^{(1,2)} + (1-\nu) w^{(1,2)}\right)$$

$$= -D\left(w^{(3,0)} + w^{(1,2)}\right)$$

$$Q_2 = \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2}$$

$$= -D\left((1-\nu) w^{(2,1)} + w^{(0,3)} + \nu w^{(2,1)}\right)$$

$$= -D\left(w^{(0,3)} + w^{(2,1)}\right)$$

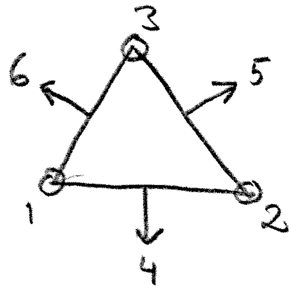
$$-f = \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2}$$

$$= -D\left(w^{(4,0)} + w^{(2,2)} + w^{(0,4)} + w^{(2,2)}\right)$$

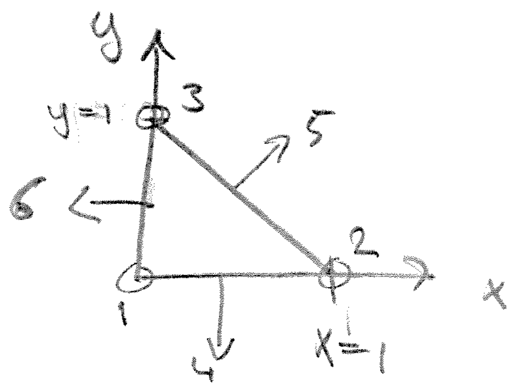
$$\Leftrightarrow D\left(\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4}\right) = f$$

Problem 2

Morley element, $w \in P_2(K) \quad \forall K$



Reference element:



Arbitrary $p \in P_2(K)$

$$p = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 y^2 + c_5 xy$$

$$\nabla p = \begin{bmatrix} c_1 + 2c_3 x + c_5 y \\ c_2 + 2c_4 y + c_5 x \end{bmatrix}$$

$$n_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad n_5 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad n_6 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\nabla p(\tfrac{1}{2}, 0) \cdot n_4 = \begin{bmatrix} c_1 + c_3 \\ c_2 + \frac{1}{2} c_5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -c_2 - \tfrac{1}{2} c_5$$

$$\nabla p(\tfrac{1}{2}, \tfrac{1}{2}) \cdot n_5 = \begin{bmatrix} c_1 + c_3 + \frac{1}{2} c_5 \\ c_2 + c_4 + \frac{1}{2} c_5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} (c_1 + c_2 + c_3 + c_4 + c_5)$$

$$\nabla p(0,\tfrac{1}{2})\cdot n_6 = \begin{bmatrix} c_1 + \frac{1}{2}c_5 \\ c_2 + c_4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -c_1 - \tfrac{1}{2}c_5$$

Vertices:

$$p(0,0) = c_0$$

$$p(1,0) = c_0 + c_1 + c_3$$

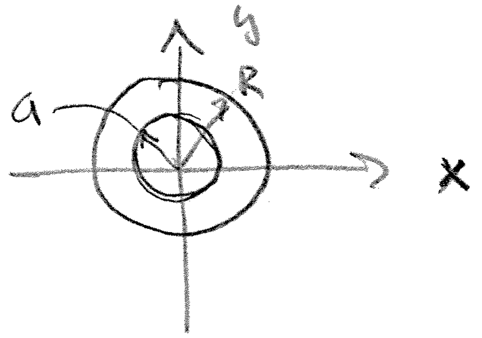
$$p(0,1) = c_0 + c_2 + c_4$$

If we give values to dofs, we should be able to solve coefc c_i i.e.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}}_{=:A} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{vertices} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{normal derivates} \end{array}$$

Clearly $\mathrm{Det}(A) \neq 0$ which proves that Morley element is unisolvent.

Problem 3



Our problem is

$$w^{(4,0)} + 2w^{(2,2)} + w^{(0,4)} = f \quad \Omega$$

Boundary conditions are

A) $w = 0$, $\frac{\partial w}{\partial n} = 0$ "clamped"

B) $w = 0$, $M_{nn} = 0$ "simple support"

In polar coordinates (r, φ)

$$\Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}$$

$$= \underbrace{\frac{1}{r}\frac{\partial w}{\partial r}}_{=\mathrm{I}} + \underbrace{\frac{\partial^2 w}{\partial r^2}}_{=\mathrm{II}} + \underbrace{\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}}_{=\mathrm{III}}$$

Our problem is

$$\Delta^2 w = f$$

$$\Delta_{\mathrm{I}}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)$$

$$= \frac{1}{r}\left(-\frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{\partial^3 w}{\partial r^3} - \frac{2}{r^3}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^3 w}{\partial r\partial\varphi^2}\right.$$

$$= -\frac{1}{r^3}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial^3 w}{\partial r^3} - \frac{2}{r^4}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r^3}\frac{\partial^3 w}{\partial r\partial\varphi^2}$$

$$\Delta_{\underline{\mathrm{II}}}^2 = \frac{\partial}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)$$

$$= \frac{\partial}{\partial r}\left(-\frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{\partial^3 w}{\partial r^3} - \frac{2}{r^3}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^3 w}{\partial r\partial\varphi^2}\right)$$

$$= \frac{2}{r^3}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial^3 w}{\partial r^3} + \frac{\partial^4 w}{\partial r^4}$$

$$+ \frac{6}{r^4}\frac{\partial^2 w}{\partial \varphi^2} - \frac{2}{r^3}\frac{\partial^3 w}{\partial r\partial\varphi^2} - \frac{2}{r^3}\frac{\partial^3 w}{\partial r\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^4 w}{\partial r^2\partial\varphi^2}$$

$$= \frac{2}{r^3}\frac{\partial w}{\partial r} - \frac{2}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial^3 w}{\partial r^3} + \frac{\partial^4 w}{\partial r^4}$$

$$+ \frac{6}{r^4}\frac{\partial^2 w}{\partial \varphi^2} - \frac{4}{r^3}\frac{\partial^3 w}{\partial r\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^4 w}{\partial r^2\partial\varphi^2}$$

$$\Delta_{\underline{\mathrm{III}}}^2 = \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)$$

$$= \frac{1}{r^3}\frac{\partial^3 w}{\partial r\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^4 w}{\partial r^2\partial\varphi^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \varphi^4}$$

$$\Delta^2 w = \Delta^2_{I} w + \Delta^2_{II} w + \Delta^2_{III} w$$

$$= -\frac{1}{r^3}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial^3 w}{\partial r^3} - \frac{2}{r^4}\frac{\partial^2 w}{\partial \varphi^2} + \frac{1}{r^3}\frac{\partial^3 w}{\partial r \partial \varphi^2}$$

$$+ \frac{2}{r^3}\frac{\partial w}{\partial r} - \frac{2}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial^3 w}{\partial r^3} + \frac{\partial^4 w}{\partial r^4}$$

$$+ \frac{6}{r^4}\frac{\partial^2 w}{\partial \varphi^2} - \frac{4}{r^3}\frac{\partial^3 w}{\partial r \partial \varphi^2} + \frac{1}{r^2}\frac{\partial^4 w}{\partial r^2 \partial \varphi^2}$$

$$+ \frac{1}{r^3}\frac{\partial^3 w}{\partial r \partial \varphi^2} + \frac{1}{r^2}\frac{\partial^4 w}{\partial r^2 \partial \varphi^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \varphi^4}$$

$$= \frac{1}{r^3}\frac{\partial w}{\partial r} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} + \frac{4}{r^4}\frac{\partial^2 w}{\partial \varphi^2}$$

$$- \frac{2}{r^3}\frac{\partial^3 w}{\partial r \partial \varphi^2} + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2 \partial \varphi^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \varphi^4}$$

$$+ \frac{\partial^4 w}{\partial r^4}$$

It seems reasonable to assume that $w(r, \varphi) = w(r)$ since both load and boundary conditions are constant along φ-axis.

$$\Delta^2 w(r) = \frac{1}{r^3} w' - \frac{1}{r^2} w'' + \frac{2}{r} w''' + w''''$$

What about the boundary conditions?

At $r=0$, we must have

$w' = 0$ (no kink)

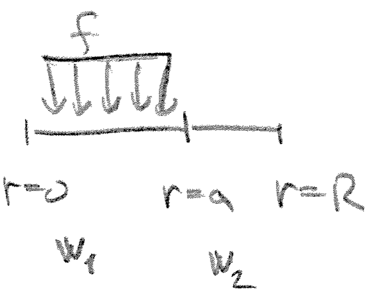
$w''' = 0$ (no shear !)

At outer boundary, i.e. $r=R$, we have

$w=0$ and $w'=0$ for clamped

$w'' = 0$ for simple supp.

We also need to consider compatibility conditions at $r=a$ since



At $r=a$ we have

$w_1 = w_2$

$w_1' = w_2'$ (no kink)

$w_1'' = w_2''$ (no jump in moment)

$w_1''' = w_2'''$

Solution to

$$w'''' + \frac{2}{r} w''' - \frac{1}{r^2} w'' + \frac{1}{r^3} w' = f$$

is

$$w(r) = C_4 + C_3 \left(\frac{1}{2} r^2 \ln(r) - \frac{1}{4} r^2 \right)$$

$$+ C_2 \frac{1}{2} r^2 + C_1 \ln(r) + \frac{f r^4}{64}$$

At $r \in (0, a)$ we have $w_1(r)$.

1) $w_1(0) < \infty \Rightarrow C_1 = 0$

2) $w'(r) = C_3 \, r \ln(r) + C_2 r + \frac{f r^3}{16}$

$\Rightarrow w'(0) = 0$

3) $w'''(r) = \frac{C_3}{r} + \frac{3 f r}{8}$

$\Rightarrow C_3 = 0$

$$w_1(r) = C_4 + C_2 \frac{1}{2} r^2 + \frac{f r^4}{64}$$

At $r \in (a, R)$ we have $w_2(r)$.

$$w_2(r) = D_4 + D_3\left(\frac{1}{2}r^2\ln(r) - \frac{1}{4}r^2\right)$$
$$+ D_2\frac{1}{2}r^2 + D_1\ln(r) + \underbrace{\frac{fr^4}{64}}_{\substack{=0 \text{ since} \\ \text{here } f=0.}}$$

Solve $C_2, C_4, D_1, D_2, D_3, D_4$

from

$$w_2(R) = 0$$
$$w_2'(R) = 0$$
$$w_1(a) = w_2(a) \qquad w_1''(a) = w_2''(a)$$
$$w_1'(a) = w_2'(a) \qquad w_1'''(a) = w_2'''(a)$$

In[135]:=

```
Rlaplace[w_] := 1 / r * D[r * D[w, r], r] + 1 / r ^ 2 * D[w, φ, φ]
```

Define Δ-oper.

In[136]:=

```
yht = Expand[Rlaplace[Rlaplace[w[r]]]]
```

diff eq.

Out[136]=

$$\frac{w'[r]}{r^3} - \frac{w''[r]}{r^2} + \frac{2\,w^{(3)}[r]}{r} + w^{(4)}[r]$$

In[137]:=

```
dsol = DSolve[yht == f, w, r]
```

Out[137]=

$$\left\{\left\{w \to \text{Function}\left[\{r\},\ \frac{f\,r^4}{64} + \frac{1}{2}\,r^2\,C[2] - \frac{1}{4}\,r^2\,C[3] + C[4] + C[1]\,\text{Log}[r] + \frac{1}{2}\,r^2\,C[3]\,\text{Log}[r]\right]\right\}\right\}$$

In[138]:=

```
ww = dsol[[1, 1, 2]]
```

solution to diffeq.

Out[138]=

$$\text{Function}\left[\{r\},\ \frac{f\,r^4}{64} + \frac{1}{2}\,r^2\,C[2] - \frac{1}{4}\,r^2\,C[3] + C[4] + C[1]\,\text{Log}[r] + \frac{1}{2}\,r^2\,C[3]\,\text{Log}[r]\right]$$

In[109]:=

```
w1[r_] = ww[r] /. {C[3] → 0, C[1] → 0}
w2[r_] = ww[r] /. {C[1] → D1, C[2] → D2, C[3] → D3, C[4] → D4, f → 0}
```

Set some bc's

Out[109]=

$$\frac{f\,r^4}{64} + \frac{1}{2}\,r^2\,C[2] + C[4]$$

Out[110]=

$$D4 + \frac{D2\,r^2}{2} - \frac{D3\,r^2}{4} + D1\,\text{Log}[r] + \frac{1}{2}\,D3\,r^2\,\text{Log}[r]$$

In[111]:=

```
ratk = Solve[{w2[R] == 0, w2'[R] == 0,
   w1[a] == w2[a], w1'[a] == w2'[a],
   w1''[a] == w2''[a], w1'''[a] == w2'''[a]}, {C[2], C[4], D1, D2, D3, D4}]
```

solve rest of coefs

Out[111]=

$$\left\{\left\{C[4] \to \frac{1}{64}\,(-3\,a^4\,f + 4\,a^2\,f\,R^2 + 4\,a^4\,f\,\text{Log}[a] - 4\,a^4\,f\,\text{Log}[R]),\right.\right.$$
$$D4 \to \frac{1}{32}\,(a^4\,f + 2\,a^2\,f\,R^2 - 2\,a^4\,f\,\text{Log}[R]),\ C[2] \to -\frac{a^4\,f - 4\,a^2\,f\,R^2\,\text{Log}[a] + 4\,a^2\,f\,R^2\,\text{Log}[R]}{16\,R^2},$$
$$\left.\left.D2 \to -\frac{a^4\,f + 4\,a^2\,f\,R^2\,\text{Log}[R]}{16\,R^2},\ D1 \to \frac{a^4\,f}{16},\ D3 \to \frac{a^2\,f}{4}\right\}\right\}$$

In[112]:=

```
sol1[r_] = w1[r] /. ratk[[1]]
sol2[r_] = w2[r] /. ratk[[1]]
```

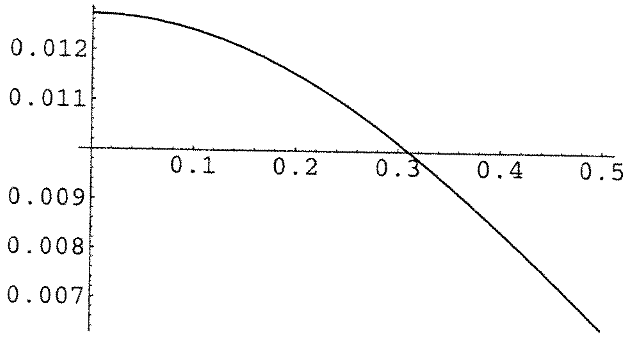
Out[112]=

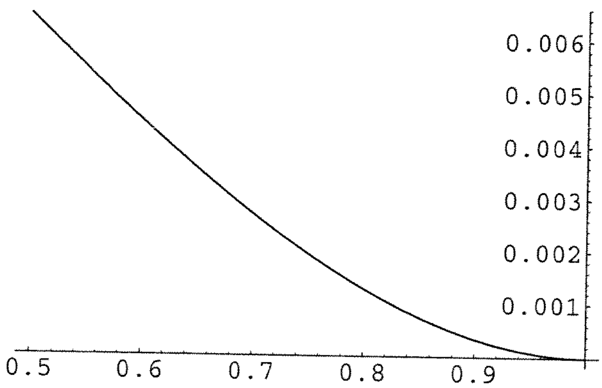
$$\frac{f\,r^4}{64} + \frac{1}{64}\,(-3\,a^4\,f + 4\,a^2\,f\,R^2 + 4\,a^4\,f\,\text{Log}[a] - 4\,a^4\,f\,\text{Log}[R]) - \frac{r^2\,(a^4\,f - 4\,a^2\,f\,R^2\,\text{Log}[a] + 4\,a^2\,f\,R^2\,\text{Log}[R])}{32\,R^2}$$

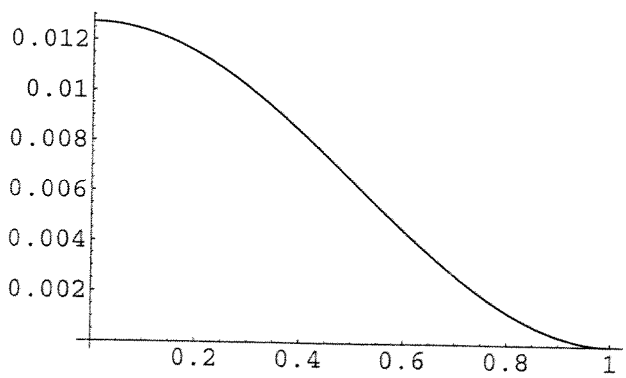
Out[113]=

$$-\frac{1}{16}\,a^2\,f\,r^2 + \frac{1}{16}\,a^4\,f\,\text{Log}[r] + \frac{1}{8}\,a^2\,f\,r^2\,\text{Log}[r] + \frac{1}{32}\,(a^4\,f + 2\,a^2\,f\,R^2 - 2\,a^4\,f\,\text{Log}[R]) - \frac{r^2\,(a^4\,f + 4\,a^2\,f\,R^2\,\text{Log}[R])}{32\,R^2}$$

In[114]:=
```
aa = 1 / 2;
RR = 1;
ff = 1 / π / aa^2;
kuva1 = Plot[sol1[r] /. {f → ff, a → aa, R → RR}, {r, 0, aa}];
kuva2 = Plot[sol2[r] /. {f → ff, a → aa, R → RR}, {r, aa, RR}];
Show[kuva1, kuva2]
```







Out[119]=
- Graphics -

In[120]:=

```
lsol1[r_] = Limit[sol1[r] /. {f → 1/π/a^2}, a → 0]
lsol2[r_] = Limit[sol2[r] /. {f → 1/π/a^2}, a → 0]
```

limit study

Out[120]=

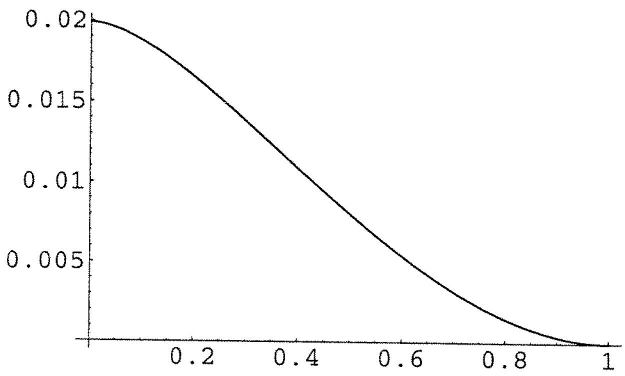
DirectedInfinity[Sign[r]^4]

Out[121]=

$$\frac{-r^2 + R^2 + 2\,r^2\,\mathrm{Log}[r] - 2\,r^2\,\mathrm{Log}[R]}{16\,\pi}$$

In[122]:=

```
Plot[lsol2[r] /. {R → 1}, {r, 0, 1}]
```



Out[122]=

- Graphics -

same for simple support

In[123]:=

```
ratk = Solve[{w2[R] == 0, w2''[R] == 0,
   w1[a] == w2[a], w1'[a] == w2'[a],
   w1''[a] == w2''[a], w1'''[a] == w2'''[a]}, {C[2], C[4], D1, D2, D3, D4}]
```

Out[123]=

$$\Big\{\Big\{C[4] \to \frac{1}{64}\,(-7\,a^4\,f + 12\,a^2\,f\,R^2 + 4\,a^4\,f\,\mathrm{Log}[a] - 4\,a^4\,f\,\mathrm{Log}[R]),$$

$$D4 \to \frac{1}{32}\,(-a^4\,f + 6\,a^2\,f\,R^2 - 2\,a^4\,f\,\mathrm{Log}[R]),$$

$$C[2] \to -\frac{a^2\,f}{4} + \frac{a^4\,f}{16\,R^2} + \frac{1}{4}\,a^2\,f\,\mathrm{Log}[a] - \frac{1}{4}\,a^2\,f\,\mathrm{Log}[R],$$

$$D2 \to -\frac{-a^4\,f + 4\,a^2\,f\,R^2 + 4\,a^2\,f\,R^2\,\mathrm{Log}[R]}{16\,R^2},\ D1 \to \frac{a^4\,f}{16},\ D3 \to \frac{a^2\,f}{4}\Big\}\Big\}$$

In[124]:=

```
sol1[r_] = w1[r] /. ratk[[1]]
sol2[r_] = w2[r] /. ratk[[1]]
```

Out[124]=

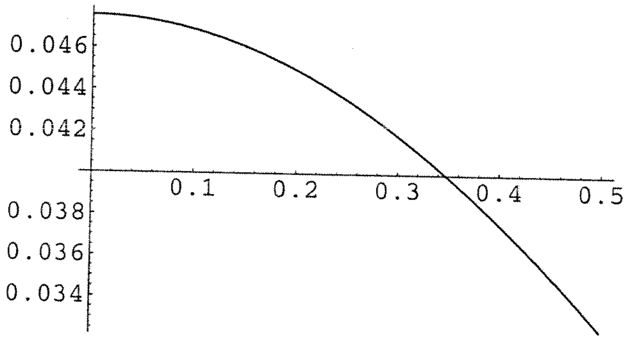
$$\frac{f\,r^4}{64} + \frac{1}{2} r^2 \left(-\frac{a^2 f}{4} + \frac{a^4 f}{16 R^2} + \frac{1}{4} a^2 f \,\mathrm{Log}[a] - \frac{1}{4} a^2 f \,\mathrm{Log}[R]\right) + \frac{1}{64} \left(-7 a^4 f + 12 a^2 f R^2 + 4 a^4 f \,\mathrm{Log}[a] - 4 a^4 f \,\mathrm{Log}[R]\right)$$

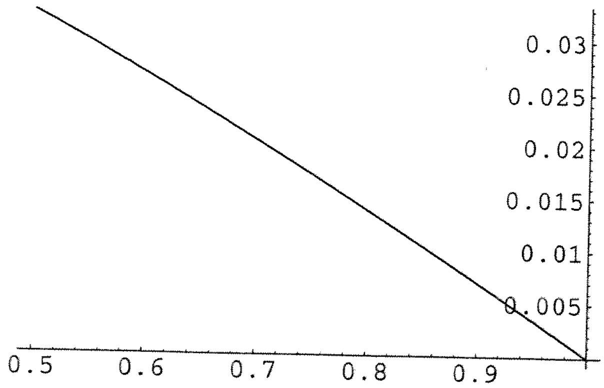
Out[125]=

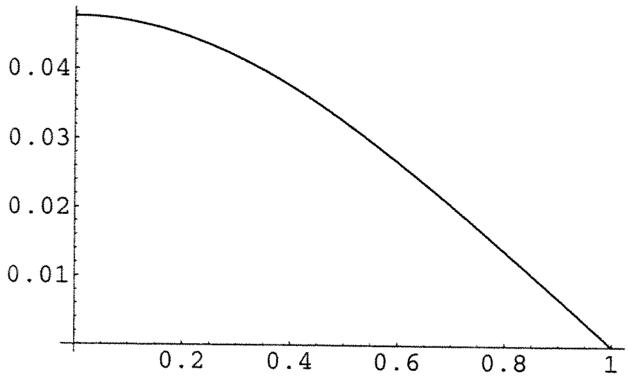
$$-\frac{1}{16} a^2 f r^2 + \frac{1}{16} a^4 f \,\mathrm{Log}[r] + \frac{1}{8} a^2 f r^2 \,\mathrm{Log}[r] + \frac{1}{32} \left(-a^4 f + 6 a^2 f R^2 - 2 a^4 f \,\mathrm{Log}[R]\right) - \frac{r^2 \left(-a^4 f + 4 a^2 f R^2 + 4 a^2 f R^2 \,\mathrm{Log}[R]\right)}{32 R^2}$$

In[126]:=

```
aa = 1 / 2;
RR = 1;
ff = 1 / π / aa^2;
kuva1 = Plot[sol1[r] /. {f → ff, a → aa, R → RR}, {r, 0, aa}];
kuva2 = Plot[sol2[r] /. {f → ff, a → aa, R → RR}, {r, aa, RR}];
Show[kuva1, kuva2]
```







Out[131]=

- Graphics -

In[132]:=

```
lsol1[r_] = Limit[sol1[r] /. {f → 1 / π / a^2}, a → 0]
lsol2[r_] = Limit[sol2[r] /. {f → 1 / π / a^2}, a → 0]
```

Out[132]=

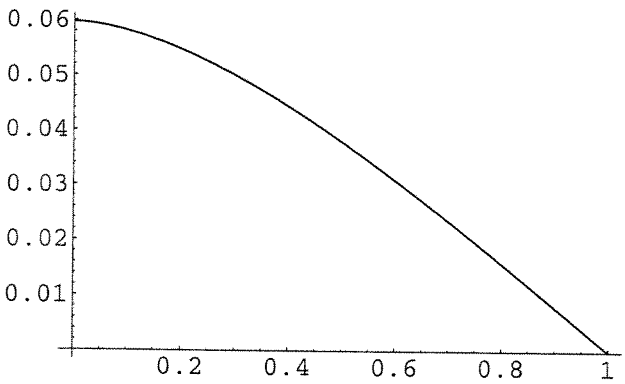
DirectedInfinity[Sign[r]^4]

Out[133]=

$$\frac{-3\, r^2 + 3\, R^2 + 2\, r^2\, \mathrm{Log}[r] - 2\, r^2\, \mathrm{Log}[R]}{16\, \pi}$$

In[134]:=

```
Plot[lsol2[r] /. {R → 1}, {r, 0, 1}]
```



Out[134]=

- Graphics -