Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**

Stenberg/Juntunen

Exercise 5

Problem 1

How does the strain tensor $\mathbf{E}$,

$$\mathbf{E}_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\Big),$$

change under an orthogonal chance of coordinates $\mathbf{x}' = \mathbf{A}\mathbf{x}$? How is the case for the infinitesimal strain tensor $\boldsymbol{\varepsilon}$

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

Problem 2 **(home exercise)**

a) Show that the physical meaning of $\sum_{k=1}^{3} \varepsilon_{kk} = \mathrm{tr}(\boldsymbol{\varepsilon})$ is the relative change in volume.

b) Assume that body is under hydrostatic pressure

$$\boldsymbol{\sigma} = -p\boldsymbol{I}, \qquad p = \text{constant} > 0$$

and show that

$$\sum_{k=1}^{3} \varepsilon_{kk} = -\frac{1}{\kappa}p, \qquad \kappa = \lambda + \frac{2}{3}\mu = \text{bulk modulus}.$$

c) Hence, the condition $\kappa > 0$ is natural. Present κ in terms of Youngs modulus E and the Poisson ratio ν. What is the condition for ν?

Problem 3

Let the characteristic polynomial for $\boldsymbol{\sigma}$ be

$$\det(\boldsymbol{\sigma} - \alpha\boldsymbol{I}) = -\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3.$$

Show that the invariants are

$$\begin{aligned}
I_1 &= \sum_{k=1}^{3} \sigma_{kk} = \mathrm{tr}(\boldsymbol{\sigma}),\\
I_2 &= \sum_{i,j=1}^{3} \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})\\
&= \frac{1}{2}[(\mathrm{tr}(\boldsymbol{\sigma}))^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)]\\
&= \begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{vmatrix} + \begin{vmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{13} & \sigma_{33}\end{vmatrix} + \begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{vmatrix},\\
I_3 &= \det(\boldsymbol{\sigma}).
\end{aligned}$$

Show that the invariants do not change under orthogonal coordinate transformations.

Problem 4

Let the strain state be a ”pure shear”, e.g.

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compute the principal stresses and directions.