Maf - 5.156 Theory of Elasticity

These notes are based on material from the following sources:

I. H. Shames. Solid Mechanics. A variational Approach. McGraw-Hill 1973

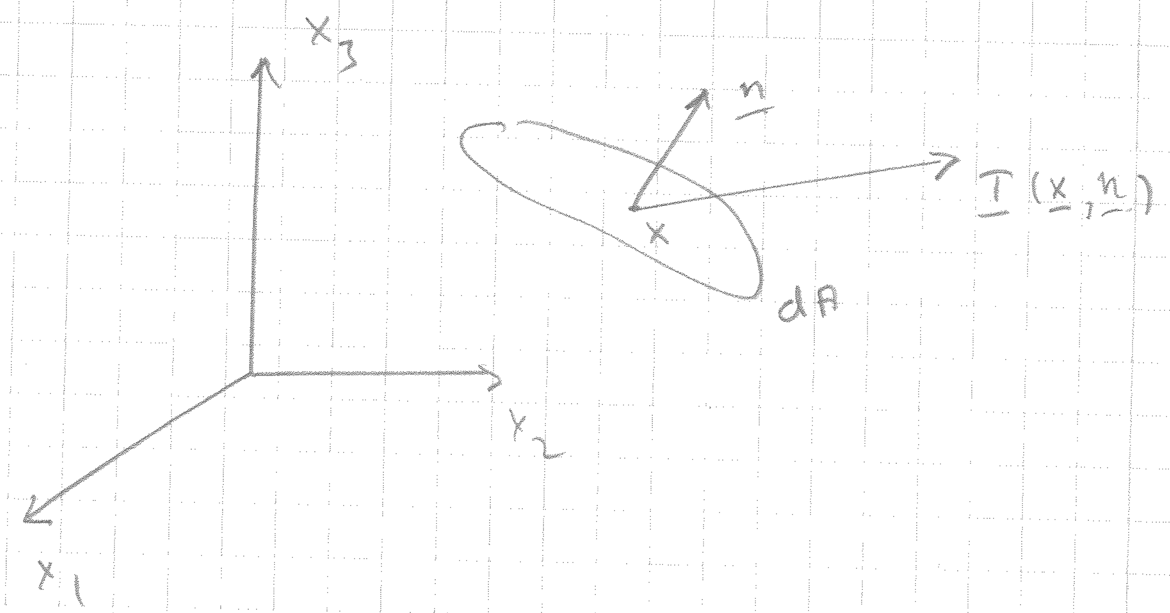
J. Nečas & I. Hlaváček. Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction. Elsevier 1981

D. Morgenstern & I. Szabó. Vorlesungen über Theoretische Mechanik. Springer-Grundlehren, Band 112, 1961.

1. The linear Theory of Elasticity

1.1. Force and Stress

The stress vector



$\underline{n}$ = the unit normal to dA

The force that acts on dA is $\approx \underline{T}(\underline{x}, \underline{n})\, dA$.

Let $\underline{T} = (T_1, T_2, T_3)$ and denote (Here we "transpose" the classical notation).

$$T_i(\underline{x}, \underline{e}_j) = \sigma_{ij} \quad , i,j = 1,2,3,$$

The numbers σ_{ij} form the <u>stress tensor</u> (matrix)

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

We have

$$\underline{T}(\underline{x}, \underline{e}_1) = \sum_{i=1}^{3} T_i(\underline{x}, \underline{e}_1)\, \underline{e}_i$$

$$= \sigma_{11}\underline{e}_1 + \sigma_{21}\underline{e}_2 + \sigma_{31}\underline{e}_3.$$

and

$$\underline{T}(\underline{x}, \underline{e}_2) = \sigma_{12}\, \underline{e}_1 + \sigma_{22}\, \underline{e}_2 + \sigma_{32}\, \underline{e}_3 .$$

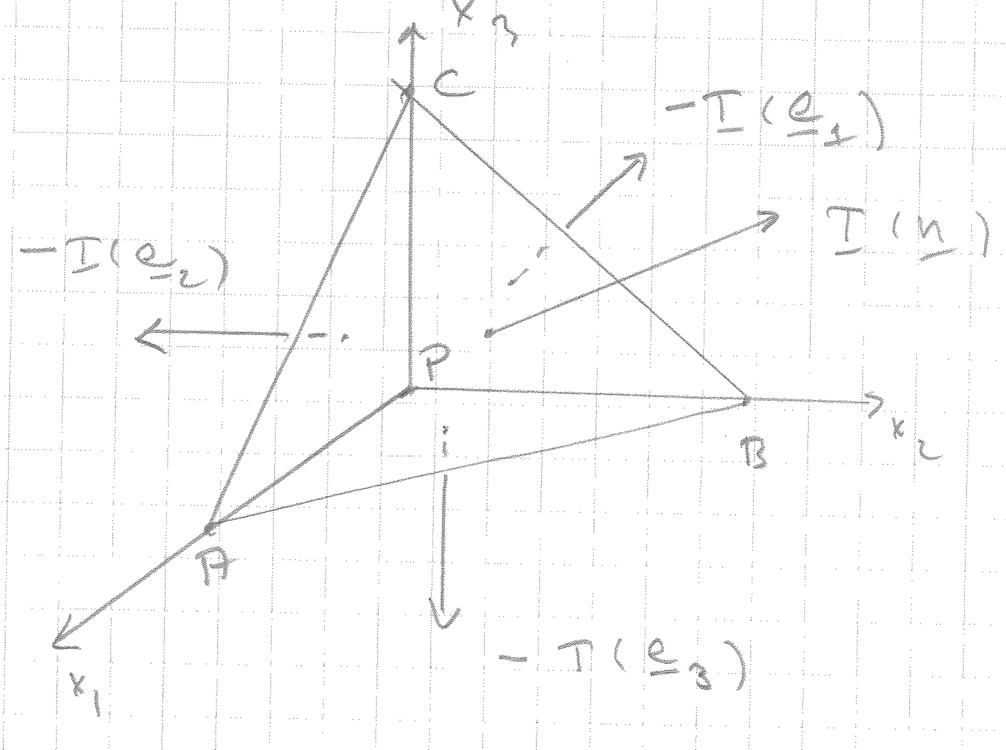
$$\underline{T}(\underline{x}, \underline{e}_3) = \sigma_{13}\, \underline{e}_1 + \sigma_{23}\, \underline{e}_2 + \sigma_{33}\, \underline{e}_3 .$$

We now prove the basic theorem (Cauchy 1822).

Theorem. The stress tensor uniquely determines the stress vector. It holds

$$\underline{T}(\underline{x}, \underline{n}) = \underline{\underline{\sigma}}\, \underline{n} .$$

Proof. We consider the "Cauchy tetrahedron" Δ:



By Newtons 3rd Law it holds $\underline{T}(\underline{x},\underline{n}) = -\underline{T}(\underline{x},-\underline{n})$.

Let $\underline{F}\,(=\underline{f}-\rho\dot{\underline{v}})$ be the volume forces action on Δ, that is body forces and inertial forces. Let h be the height of Δ, base ABC $= dA$. The other surfaces have the areas $dA\, n_i$, $\underline{n} = (n_1, n_2, n_3)$. Let us write down the balance of forces.

$$-\underline{T}(\underline{e}_1)\,dA_1 - \underline{T}(\underline{e}_2)\,dA_2 - \underline{T}(\underline{e}_3)\,dA_3$$

$$+\ \underline{T}(\underline{n})\,dA = \tfrac{1}{3}\,h\,dA\,F.$$

Substitute $dA_i = dA\,n_i$ and divide with dA $\Longrightarrow$

$$-\underline{T}(\underline{e}_1)n_1 - \underline{T}(\underline{e}_2)n_2 - \underline{T}(\underline{e}_3)n_3$$
$$+ \underline{T}(\underline{n}) = \frac{h}{3}\underline{F}$$

Let $h \to 0$ and assume, that $\underline{F}$ is bounded. This gives

$$\underline{T}(\underline{n}) = \underline{T}(\underline{e}_1)n_1 + \underline{T}(\underline{e}_2)n_2 + \underline{T}(\underline{e}_3)n_3$$

$$= \sum_{j=1}^{3} \underline{T}(\underline{e}_j)\, n_j$$

$$= \sum_{i=1}^{3}\Big(\sum_{j=1}^{3} \sigma_{ij} n_j\Big)\underline{e}_i$$

Hence, we note that

$$\underline{T}(\underline{n}) = \underline{\underline{\sigma}}\,\underline{n}. \qquad \square.$$

Using the Einstein convention we write

$$T_i = \sigma_{ij} n_j .$$

1.2. Partial integration in $\mathbb{R}^3$.

(Green's formulas)

The basic formula. $f : D \to \mathbb{R}$

$$f = f(\underline{x}) = f(x_1, x_2, x_3).$$

$$\int_D \frac{\partial f}{\partial x_i}\, dV = \int_{\partial D} f\, \underline{e}_i \cdot \underline{n}\, dS$$

Using Einstein's notation:

$$\frac{\partial f}{\partial x_i} = f_{,i}$$

we have

$$\int_D f_{,i}\, dV = \int_{\partial D} f\, n_i\, dS.$$

With this notation Gauss Divergence theorem is

$$\int_D f_{i,i}\, dV = \int_{\partial D} f_i\, n_i\, dS$$

That is $\underline{f} = (f_1, f_2, f_3)$

$$\int_D \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} \, dV = \int_{\partial D} \sum_{i=1}^{3} f_i n_i \, dS$$

or

$$\int_D \nabla \cdot \underline{f} \, dV = \int_{\partial D} \underline{f} \cdot \underline{n} \, dS.$$

We can now derive Cauchy's equations of motion. We have a body occupying the region $\Omega \subset \mathbb{R}^3$. Let $\underline{\underline{\sigma}}(\underline{x})$ be the stress field, $\underline{f}(\underline{x})$ the body force an $\underline{v}(\underline{x})$ the velocity field. Then it holds:

$$\left.\begin{aligned} \sigma_{ij,j} + f_i &= \rho \dot{v}_i \\ \sigma_{ij} &= \sigma_{ji} \end{aligned}\right\} \text{ in } \Omega.$$

We define the divergence of a tensor $\underline{\underline{\tau}}$ as the vector $\operatorname{div} \underline{\underline{\tau}}$ given by

$$(\operatorname{div} \underline{\underline{\tau}})_i = \sum_{j=1}^{3} \tau_{ij,j} .$$

Cauchy's eqs. can then be written as

$$\left.\begin{aligned} \operatorname{div} \underline{\underline{\sigma}} + \underline{f} &= \rho \underline{\dot{v}} \\ \underline{\underline{\sigma}}^T &= \underline{\underline{\sigma}} \end{aligned}\right\} \text{ in } \Omega .$$

Derivation. Let $D \subset \Omega$ be arbitrary. The force acting on D is

$$\int_{\partial D} \underline{T}(\underline{n}) \, dS + \int_D \underline{f} \, dV$$

Substitute $\underline{T}(\underline{n}) = \underline{\underline{\sigma}} \, \underline{n}$:

$$\int_{\partial D} \underline{\underline{\sigma}} \, \underline{n} \, dS + \int_D \underline{f} \, dV$$

this is equal to

$$\int_D \varrho \dot{\underline{v}}\, dV \quad , \quad \text{viz.}$$

$$\int_{\partial D} \underline{\underline{\sigma}}\, \underline{n}\, dS' + \int_D \underline{f}\, dV = \int_D \varrho \dot{\underline{v}}\, dV$$

Let's look at $\int_{\partial D} \underline{\underline{\sigma}}\, \underline{n}\, dS'$

In component form this is

$$\int_{\partial D} \sigma_{ij}\, n_j\, dS$$

By Gauss divergence theorem we have

$$\int_{\partial D} \sigma_{ij}\, n_j\, dS' = \int_D \sigma_{ij,j}\, dV,$$

that is

$$\int_{\partial D} \underline{\underline{\sigma}}\, \underline{n}\, dS' = \int_D \operatorname{div} \underline{\underline{\sigma}}\, dV.$$

For an arbitrary $D \subset \Omega$ we hence have

$$\int_D [\operatorname{div} \underline{\underline{\sigma}} + \underline{f} - \rho \underline{\dot{v}}] \, dV = 0 .$$

$$\Rightarrow \qquad \operatorname{div} \underline{\underline{\sigma}} + \underline{f} = \rho \underline{\dot{v}} .$$

Next, the relationship $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$.
The momentum balance is

$$\int_D \underline{x} \times \rho \underline{\dot{v}} \, dV = \int_D \underline{x} \times \underline{f} \, dV$$

$$+ \int_{\partial D} \underline{x} \times \underline{T}(\underline{n}) \, dS .$$

Let us look at this in component form, say the x_1 - comp.:

$$\int (x_2 \rho \dot{v}_3 - x_3 \rho \dot{v}_2) \, dV$$

$$= \int_D (x_2 f_3 - x_3 f_2) \, dV$$

$$+ \int_{\partial D} [x_2 T_3(\underline{n}) - x_3 T_2(\underline{n})] \, dS$$

The last of these is

$$\int_{\partial D} (x_2 T_3 - x_3 T_2)\, dS$$

$$= \int_{\partial D} (x_2 \sigma_{3j} n_j - x_3 \sigma_{2j} n_j)\, dV,$$

$$= \int_D ([x_2 \sigma_{3j}]_{,j} - [x_3 \sigma_{2j}]_{,j})\, dV$$

$$= \int_D (\delta_{2j} \sigma_{3j} + x_2 \sigma_{3j,j} - \delta_{3j} \sigma_{2j} - x_3 \sigma_{2j,j})\, d$$

$$= \int_D (\sigma_{32} - \sigma_{23})\, dV + \int_D (x_2 \sigma_{3j,j} - x_3 \sigma_{2j,j})\, d$$

Hence, the first component of the momentum equation is

$$\int_D (x_2 \rho \dot{v}_3 - x_3 \rho \dot{v}_2)\, dV$$

$$= \int_D (x_2 f_3 - x_3 f_2)\, dV + \int_D (x_2 \sigma_{3j,j} - x_3 \sigma_{2j,j})\, d$$

$$+ \int_D (\sigma_{32} - \sigma_{23})\, dV .$$

Recalling the equilibrium equations

$$\rho \dot{v}_3 = \sigma_{3j,j} + f_3$$

$$\rho \dot{v}_2 = \sigma_{2j,j} + f_2 .$$

We see that

$$\int_D (\sigma_{32} - \sigma_{23})\, dV = 0 \qquad \forall D \subset \Omega.$$

Hence, $\sigma_{32} = \sigma_{23}$.

Similarly $\sigma_{13} = \sigma_{31}$ and $\sigma_{12} = \sigma_{21}$

$\square$.

Transformation of coordinates

Two orthogonal basis: $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\underline{e}'_1, \underline{e}'_2, \underline{e}'_3)$, they are related by a rotation. We denote (E's convention)

$$\underline{e}'_i = a_{ij}\underline{e}_j \quad \left(= \sum_{j=1}^{3} a_{ij}\underline{e}_j\right)$$

Let $\underline{A} = \{a_{ij}\}$. The matrix is orthogonal

$$\underline{A}^{-1} = \underline{A}^T$$

and it holds

$$\underline{e}_i = a_{ji}\underline{e}'_j .$$

Let $\vec{v}$ be a vector:

$$\vec{v} = x_i\underline{e}_i = x'_j\underline{e}'_j .$$

For the components we get the transformation formulas

$$x'_k = \vec{v}\cdot\underline{e}'_k = \vec{v}\cdot(a_{kj}\underline{e}_j)$$

$$= a_{kj}\,\vec{v}\cdot\underline{e}_j = a_{kj}x_j$$

That is

$$x'_i = a_{ij} x_j$$

or

$$\underline{x}' = \underline{A}\,\underline{x} .$$

Next, we write down the transformation formulas for the stress tensor.

We have

$$\sigma_{ij} = \underline{T}(\underline{e}_j) \cdot \underline{e}_i = e_i \cdot \underline{T}(\underline{e}_j)$$

and

$$\sigma'_{kl} = \underline{e}'_k \cdot \underline{T}(\underline{e}'_l) .$$

This gives

$$\sigma'_{kl} = a_{ki} \underline{e}_i \cdot \underline{T}(a_{lj} \underline{e}_j)$$

$$= a_{ki} a_{lj} \underline{e}_i \cdot \underline{T}(\underline{e}_j)$$

$$= a_{ki} a_{lj} \sigma_{ij} .$$

That is:

$$\sigma'_{kl} = a_{ki} a_{lj} \sigma_{ij} .$$

In matrix language

$$\sigma'_{kl} = a_{ki}\,\sigma_{ij}\,a_{lj}$$

gives

$$\underline{\underline{\sigma}}' = \underline{\underline{A}}\,\underline{\underline{\sigma}}\,\underline{\underline{A}}^T ,$$

The same derivation

$$\underline{T}' = \underline{\underline{A}}\,\underline{T} \quad , \quad \underline{n}' = \underline{\underline{A}}\,\underline{n}$$

and $\underline{T}' = \underline{\underline{\sigma}}'\,\underline{n}'$ & $\underline{T} = \underline{\underline{\sigma}}\,\underline{n}$

We substitute:

$$\underline{T} = \underline{\underline{A}}^T\,\underline{T}' = \underline{\underline{A}}^T\,\underline{\underline{\sigma}}'\,\underline{n}'$$

$$= \underline{\underline{A}}^T\,\underline{\underline{\sigma}}'\,\underline{\underline{A}}\,\underline{n} \overset{\text{also}}{=} \underline{\underline{\sigma}}\,\underline{n} .$$

Hence,

$$\underline{\underline{\sigma}} = \underline{\underline{A}}^T\,\underline{\underline{\sigma}}'\,\underline{\underline{A}}$$

$$\Leftrightarrow \quad \underline{\underline{\sigma}}' = \underline{\underline{A}}\,\underline{\underline{\sigma}}\,\underline{\underline{A}}^T .$$

Principal stresses

Definition

$$\underline{\underline{\sigma}}\,\underline{n} = \sigma\,\underline{n}$$

so that the shear stress vanish.

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ gives $\Rightarrow$ there is three real principal stresses and orthogonal principal directions of stress.

The characteristic polynomial

$$\sigma^3 - I_1\,\sigma^2 + I_2\,\sigma^1 - I_3 = 0$$

with the stress invariants

$$I_1 = \sigma_{ii} = \mathrm{tr}(\underline{\underline{\sigma}}) = \sigma_{11} + \sigma_{22} + \sigma_{33},$$

$$I_2 = \frac{1}{2}\left[(\mathrm{tr}\,\underline{\underline{\sigma}})^2 - \mathrm{tr}(\underline{\underline{\sigma}}^2)\right],$$

$$= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})$$

$$= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix}$$

$$I_3 = \det(\underline{\underline{\sigma}}) .$$

In the pricipal axis system the tensor is diagonal

$$\underline{\underline{\sigma}} = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$$

from which we obtain

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \quad , \quad I_3 = \sigma_1 \sigma_2 \sigma_3$$

and

$$I_2 = (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3) . \quad \square$$

<u>Problem</u>. Find the maximum shear stresses at a point.

<u>Solution</u>. Let N be the normal stress and $\underline{S}$ the shear,

$$\underline{T} = N \underline{n} + \underline{S}$$

It holds $|\underline{T}|^2 = N^2 + |\underline{S}|^2$.

We will work in the principal axis system.

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

We assume that $\sigma_i \neq \sigma_j$ $i \neq j$.

$$\underline{T} = \underline{\underline{\sigma}}\,\underline{n} \qquad \text{gives}$$

$$T_i = \sigma_i n_i \,, \qquad \text{no summation!}$$

Hence, we get

$$|\underline{T}|^2 = \sum_{i=1}^{3} \sigma_i^2 n_i^2$$

$$N^2 = (\underline{T} \cdot \underline{n})^2 = (\underline{n}^T \underline{\underline{\sigma}}\, \underline{n})^2$$

$$= \left(\sum_{i=1}^{3} \sigma_i n_i^2 \right)^2$$

We thus have

$$|\underline{S}|^2 = \sum_{i=1}^{3} \sigma_i^2 n_i^2 - \left(\sum_{i=1}^{3} \sigma_i n_i^2 \right)^2.$$

We should maximize this under the constraint $\sum_{i=1}^{3} n_i^2 = 1$.

We use Lagrange multipliers.

$$L = |\underline{S}|^2 + \lambda\left(\sum_{i=1}^{3} n_i^2 - 1\right) = 0.$$

$$\frac{\partial L}{\partial n_i} = 0 \quad \text{gives}$$

$$2\sigma_i^2 n_i - 2\left(\sum_{i=1}^{3} \sigma_i n_i^2\right) \cdot 2\sigma_i n_i$$

$$- \lambda \cdot 2 n_i = 0.$$

$$\Rightarrow \quad n_i\left(\sigma_i^2 - 2\left(\sum_{i=1}^{3} \sigma_i n_i^2\right)\sigma_i + \lambda\right) n_i = 0$$

with $N = \sum_{i=1}^{3} \sigma_i n_i^2$ we have the three equations:

$$\begin{cases} (\lambda + \sigma_1^2 - 2N\sigma_1)\, n_1 = 0. \\ (\lambda + \sigma_2^2 - 2N\sigma_2)\, n_2 = 0. \\ (\lambda + \sigma_3^2 - 2N\sigma_3)\, n_3 = 0. \end{cases}$$

Since $\sum_{i=1}^{3} n_i^2 = 1$ at least one of the expressions in brackets have to vanish.

If only one vanish this

gives a principal direction (i.e. minimizing the shear). What if all three vanish?

$$\lambda + \sigma_1^2 - 2N\sigma_1 = 0.$$

$$\lambda + \sigma_2^2 - 2N\sigma_2 = 0.$$

$$\lambda + \sigma_3^2 - 2N\sigma_3 = 0.$$

or

$$\begin{bmatrix} 1 & \sigma_1 & \sigma_1^2 \\ 1 & \sigma_2 & \sigma_2^2 \\ 1 & \sigma_3 & \sigma_3^2 \end{bmatrix} \begin{bmatrix} \lambda \\ -2N \\ 1 \end{bmatrix} = 0.$$

$$\Rightarrow \det \begin{bmatrix} 1 & \sigma_1 & \sigma_1^2 \\ 1 & \sigma_2 & \sigma_2^2 \\ 1 & \sigma_3 & \sigma_3^2 \end{bmatrix} = 0.$$

Conclusion: Two of the brackets have to vanish, we have three alternatives. The first:

$$\lambda + \sigma_1^2 - 2N\sigma_1 = 0.$$

$$\lambda + \sigma_2^2 - 2N\sigma_2 = 0.$$

$$\sigma_1^2 - \sigma_2^2 - 2N(\sigma_1 - \sigma_2) = 0.$$

$$N = \frac{1}{2}(\sigma_1 + \sigma_2)$$

$$\lambda = (\sigma_1 + \sigma_2)\sigma_1 - \sigma_1^2 = \sigma_1\sigma_2$$

On the other hand, we have $N = \sigma_1 n_1^2 + \sigma_2 n_2^2$.

Hence

$$\sigma_1 (n_1^2 - 1/2) + \sigma_2 (n_2^2 - 1/2) = 0.$$

$$n_1^2 + n_2^2 = 1.$$

this gives

$$\sigma_1 (n_1^2 - 1/2) + \sigma_2 (\underbrace{1 - n_1^2 - 1/2}_{= 1/2 - n_1^2}) = 0.$$

$$(\sigma_1 - \sigma_2)(n_1^2 - 1/2) = 0.$$

and $\sigma_1 \neq \sigma_2$ gives

$$n_1 = n_2 = \pm \frac{\sqrt{2}}{2}.$$

Finally $|\underline{S}|$?

$$|\underline{S}|^2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2) - \left((\sigma_1 + \sigma_2)\frac{1}{2}\right)^2$$

$$= \frac{1}{2}(\sigma_1^2 + \sigma_2^2) - \frac{1}{4}(\sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2)$$

$$= \frac{1}{4}(\sigma_1 - \sigma_2)^2$$

$$|\underline{S}| = \frac{1}{2}|\sigma_1 - \sigma_2|.$$

We thus get the following directions and shear stresses:

$$\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}, 0 \right), \quad |\underline{s}| = \frac{1}{2} |\sigma_1 - \sigma_2|$$

$$\left(0, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2} \right), \quad |\underline{s}| = \frac{1}{2} |\sigma_2 - \sigma_3|$$

$$\left(\pm \frac{\sqrt{2}}{2}, 0, \pm \frac{\sqrt{2}}{2} \right), \quad |\underline{s}| = \frac{1}{2} |\sigma_1 - \sigma_3| .$$

<u>Remark</u>. This result is used for formulating the TRESCA yield criteria. $\square$