Helsinki University of Technology
Mat-5.156 Theory of Elasticity **Stenberg / Tuominen**
5. exercise on Tuesday 8th March 2005

1. Proof (directly) the principle of complementary energy: In the set of statically admissible stresses
$$\Sigma = \{\, \tau_{ij} \mid \tau_{ij,j} + f_i = 0 \text{ in } \Omega,\ \tau_{ij} n_j = t_i \text{ on } \Gamma_N \,\}$$
the real stress $\underline{\underline{\sigma}}$ (solution for elasticity problem) minimizes the complementary energy
$$\Psi(\underline{\underline{\tau}}) = \frac{1}{2}\int_\Omega a_{ijkl}\tau_{ij}\tau_{kl}\, dV - \int_{\Gamma_D} u_{0i}\tau_{ij}n_j\, dS,$$
where $\underline{u}_0$ is displacement given on Γ_D.
Denote the complementary energy norm as
$$||\underline{\underline{\tau}}||_C^2 = \int_\Omega a_{ijkl}\tau_{ij}\tau_{kl}\, dV.$$
Show that
$$\frac{1}{2}||\underline{\underline{\sigma}} - \underline{\underline{\tau}}||_C^2 = \Psi(\underline{\underline{\tau}}) - \Psi(\underline{\underline{\sigma}}).$$

2. **Betti's theorem.** Consider $(\underline{u}, \underline{\underline{\varepsilon}}, \underline{\underline{\sigma}})$ and $(\underline{\tilde{u}}, \underline{\underline{\tilde{\varepsilon}}}, \underline{\underline{\tilde{\sigma}}})$ as solutions for elasticity problem with loads $\underline{f}$ ja $\underline{\tilde{f}}$ respectively. Proof that
$$\begin{aligned}
&\int_{\partial\Omega} (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{\tilde{u}}\, dS + \int_\Omega (\underline{f} \cdot \underline{\tilde{u}})\, dV \\
&= \int_{\partial\Omega} (\underline{\underline{\tilde{\sigma}}}\,\underline{n}) \cdot \underline{u}\, dS + \int_\Omega (\underline{\tilde{f}} \cdot \underline{u})\, dV \\
&= \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{\tilde{\varepsilon}}}\, dV \\
&= \int_\Omega \underline{\underline{\tilde{\sigma}}} : \underline{\underline{\varepsilon}}\, dV.
\end{aligned}$$

3. (**Home exercise.**) Consider the linear elasticity problem in which the constitutive model is
$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}(\underline{u}),$$
and the equilibrium equations are
$$\sigma_{ij,j} + f_i = 0.$$
Write the boundary conditions and the energy formulation in the following cases:
   - Sliding support on $\Gamma_S \subset \Gamma$: the body is allowed to move in the tangent direction of the boundary Γ_S, and displacements in the normal direction are not allowed.
   - On boundary $\Gamma_C \subset \Gamma$ traction is directly proportional to displacement with proportional coefficient $\pm k$, $k > 0$. Which sign?

4. In the plane stress $\sigma_{i3} = 0$, $i = 1, 2, 3$. Show that for an isotropic material the constitutive model between stress σ_{ij}, $i, j = 1, 2$, and displacement $\underline{u} = (u_1, u_2)$ can be written as follows:
$$\sigma_{ij} = 2\mu\varepsilon_{ij}(\underline{u}) + \lambda^*\varepsilon_{kk}(\underline{u})\delta_{ij},$$
where
$$\lambda^* = \frac{2\mu\lambda}{2\mu + \lambda}.$$
Write the strain ε_{33} as an expression of ε_{11} and ε_{22}.