CHAPTER 7

Bifurcations of periodic solutions; connecting orbits.

We start this chapter with continuation and bifurcation detection of periodic solutions. The first part of this is just applying the studies of generic bifurcations of maps now to the Poincarè map. The second part consists of examining cases where the validity of the Poincarè map fails.

Then we study connecting orbits between hyperbolic equilibria. A connecting orbit is an important instance of a nontrivial invariant manifold, the manifold itself being a solution trajectory of the dynamical system. We will look at two tpyes of connecting orbits: an orbit between distinct equilibria, and one of an equilibrium to itself. These are called *heteroclinic* and *homoclinic*, respectively. Often, heteroclinic orbits play the role of separatrices between different basins of attraction; we saw an instance of this in Example 1.5.3. Homoclinic and heteroclinic orbits are typically encountered as limiting behavior of periodic orbits, as the period approaches infinity. So they can often be observed from continuation of periodic solutions.

The main part of this chapter will then discuss ways to approximate the connecting orbits numerically, consider continuation of such orbits, and finally also discuss connecting orbits between invariant manifolds other than fixed points, e.g. between periodic orbits.

1. Continuation and bifurcations for periodic orbits

Consider the system

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x},\ \boldsymbol{\alpha}), \quad (\boldsymbol{x},\ \boldsymbol{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^p, \tag{1.1}$$

together with its flow $\boldsymbol{\phi}$. We are interested in continuation and bifurcations of periodic solutions of (1.1). In agreement with our general strategy, at first we assume that there is one parameter free. We set up a system of equations which allows us to continue a branch of periodic orbits as well as to detect generic codimension 1 bifurcations along this branch. Later, by freeing another parameter, we may continue the branch of periodic orbits having these bifurcations, and so forth.

To study generic codimension 1 bifurcations of periodic orbits, consider the Poincaré map $\boldsymbol{P}$. A periodic orbit corresponds to a fixed point of $\boldsymbol{P}$. Therefore, bifurcations along a branch of periodic orbits are the same objects as bifurcations

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along a curve of fixed points of $\boldsymbol{P}$. As we know, generically this means one of the following three possibilities.

(i) One multiplier is 1: a *fold* or *saddle node* bifurcation. Two periodic orbits meet and disappear. Seen in the $(\boldsymbol{x}, \alpha)$ space the branch of periodic orbits turns backward in the α direction.
(ii) One multiplier is -1: a *flip* or *period doubling* bifurcation. The original periodic orbit changes its stability, while a periodic orbit of twice the period appears.
(iii) There is a complex conjugate pair of multipliers of unit modulus $e^{\pm i\theta}$: a *Neimark–Sacker* bifurcation. There appears an invariant curve for $\boldsymbol{P}$, hence an invariant torus for (1.1).

PROBLEM 1.1. Draw figures of each of these cases in three dimensions. Which of them can happen in two dimensions?

Now we are ready to set up continuation equations of periodic solutions of (1.1). In Chapters 5 and 6, we have already laid down all the ingredients we need to describe the continuation process. We recast the fundamental ideas in the present setting.

Begin by recalling that a periodic solution of (1.1) of period τ can be viewed as solution of the boundary value problem

$$\boldsymbol{x}' = \tau \boldsymbol{f}(\boldsymbol{x}, \alpha) \ , \quad \boldsymbol{x}(0) = \boldsymbol{x}(1) \ . \tag{1.2}$$

Now, suppose that we have a periodic orbit $\boldsymbol{x}^0$ of period τ^0, at a parameter value α^0. If this does not have multiplier one then for $\alpha = \alpha^0 + \Delta\alpha$ (1.2) will also have a periodic solution close to $\boldsymbol{x}^0$. However, to include also the fold case we proceed as we did in Chapter 6: we parametrize the sought solution, the period, as well as the parameter, $\boldsymbol{x}(s)$, $\tau(s)$, $\alpha(s)$, with respect to the arc-length variable:

$$|\dot{\boldsymbol{x}}|^2 + (\dot{\tau})^2 + (\dot{\alpha})^2 \ = \ 1 \, , \quad \text{and } \dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{ds} \, , \ \text{ etc.} \, . \tag{1.3}$$

A standard choice for the norm of $\boldsymbol{x}$ here is

$$|\boldsymbol{x}| = \Big(\int_0^1 |\boldsymbol{x}(t)|^2 \, (t) \ dt \Big)^{1/2} \ . \tag{1.4}$$

As in Chapter 6, rather than using (1.3), we replace it by the pseudo-arc-length parametrization. Therefore, assuming that an approximation $\boldsymbol{x}^0, \tau^0, \alpha^0$ of the solution is available, we require

$$\eta(\boldsymbol{x}, \tau, \alpha) := \int_0^1 (\boldsymbol{x}(t) - \boldsymbol{x}^0(t))^T \dot{\boldsymbol{x}}^0(t) dt \ + (\tau - \tau^0)\dot{\tau}^0 + (\alpha - \alpha^0)\dot{\alpha}^0 - \Delta s = 0 \ . \tag{1.5}$$

To conclude, recall that an appropriate phase condition for continuation of periodic orbits is

$$\sigma(\boldsymbol{x}) = \int_0^1 (\boldsymbol{x}(t) - \boldsymbol{x}^0(t))^T \boldsymbol{x}'(t)dt = 0\,. \tag{1.6}$$

Putting it all together, we obtain the following:

Given approximate solution $\boldsymbol{x}^0, \tau^0, \alpha^0$ and a steplength increase Δs, we seek $\boldsymbol{x}^1, \tau^1, \alpha^1$ by solving

$$\begin{cases} (\boldsymbol{x}^1)'(t) &= \tau^1 \boldsymbol{f}(\boldsymbol{x}^1, \alpha^1)\,, \quad 0 \le t \le 1 \\ \boldsymbol{x}^1(0) &= \boldsymbol{x}^1(1) \\ \sigma(\boldsymbol{x}^1) &= 0 \\ \eta(\boldsymbol{x}^1, \tau^1, \alpha^1) &= 0\,. \end{cases} \tag{1.7}$$

To get the continuation process started we need an initial approximation $\boldsymbol{x}^0,\ \alpha^0,\ \tau^0$. If we have a hint the for a given α^0 a periodic solution might exist, then a "shooting" approach is often a reasonable idea in this context: integrate (1.1) to find an approximate fixed point of the Poincarè map. At Hopf bifurcation points of (1.1) we get a guaranteed initial approximation. That is, while scanning the zero set of $\boldsymbol{f}(\boldsymbol{x}, \alpha)$, we have encountered a nondegenerate Hopf bifurcation point, then we have found a point $(\boldsymbol{x}_0, \alpha_0)$ where

$$\boldsymbol{f}(\boldsymbol{x}_0, \alpha_0) = \boldsymbol{0}\ ,\ \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x}_0, \alpha_0)(\boldsymbol{u}, \boldsymbol{v}) = [\boldsymbol{u}\ \boldsymbol{v}] \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}\ , \quad \boldsymbol{u}^T\boldsymbol{u} + \boldsymbol{v}^T\boldsymbol{v} = 1\,.$$

Now, recall the center manifold reduction ideas, see Section 3 of Chapter 6, in particular the part about Hopf bifurcation in $\mathbb{R}^n$. Then, we know that for α close to α_0, at leading order the dynamics are the same as those of the reduced problem:

$$\boldsymbol{y}(t) \ = \ \sin(\frac{2\pi}{\tau_0}t)\boldsymbol{u} \ + \cos(\frac{2\pi}{\tau_0}t)\boldsymbol{v}\ , \tag{1.8}$$

which is a simple periodic motion with period $\tau_0 = 2\pi\omega_0$. Therefore, we can use (with $s_0 = 0$)

$$\boldsymbol{x}^0(t, \Delta s) \ = \ \boldsymbol{x}_0 + \Delta s \boldsymbol{y}\,, \quad \alpha^0 = \alpha_0\,, \quad \tau^0 = \tau_0 = 2\pi/\omega_0\,. \tag{1.9}$$

With this choice, we can get started and then use (1.7) for continuation. Observe that for the first continuation step we have $\dot{\boldsymbol{x}}^0 = \boldsymbol{y}(t)$, and $\dot{\tau}^0 = \dot{\alpha}^0 = 0$, relations which are used in setting up the phase condition (1.6) and the pseudo-arc-length continuation equations (1.5).

Problem 1.2. With the notation of (1.9), show that we have

$$\begin{aligned} \boldsymbol{x}(t, \Delta s) \ &= \ \boldsymbol{x}^0(t, \Delta s) + O((\Delta s)^2)\,, \quad \text{and} \\ \alpha(\Delta s) \ &= \ \alpha^0(\Delta s) + O((\Delta s)^2)\,, \quad \tau(\Delta s) \ = \ \tau^0(\Delta s) + O((\Delta s)^2)\,. \end{aligned}$$

PROBLEM 1.3. Here you follow periodic orbits and further bifurcation period doubling orbits[1]. You must produce a few pictures and an explanation of what you observe.

We want to continue exploration of the system

$$\begin{aligned} u_1' &= u_1(1-u_1) - p_2 u_1 u_2 \\ u_2' &= -u_2/4 + p_2 u_1 u_2 - 3u_2 u_3 - p_1(1-e^{-5u_2}) \\ u_3' &= -u_3/2 + 3u_2 u_3 \,. \end{aligned}$$

First, find the basic bifurcation diagram: continue the solution $(0,0,0)$ from the value $p_1 = 0$ to 1.5, while holding $p_2 = 5.6$. Inspect the diagram by invoking `PLAUT`. You should see a branch point on the main branch, then after following the new branch (bottom half of the picture) there are two Hopf points and one more branch point; following the second half of this new branch, there are two more Hopf points. Consider this last Hopf point, and compute periodic orbits emanating from this (set `NTST=10`). You should see a label `PD`, and two labels `LP`. These are "period doubling" and "limit point". Now, switch branches at the period doubling point and compute the bifurcating branch of periodic orbits (require `NTST=20` and set `NMX=20` as well). Append the output to the previous one, and insepct it with `PLAUT`. Using "`bd`" in `PLAUT`, zoom on the region of interest –in the default $(p_1, \|\cdot\|)$ plane is $[0, 0.35] \times [0.15, 0.4]$– and save the plot. Then, plot a 3-d picture of the orbits at the period doubling points. (Use "`3d`" in `PLAUT`, select the label of the required orbits, choose the axes 2,3,4 and use the option "`d6`").

Codimension 1 bifurcation curves.

With the previous approach for continuing periodic orbits, and with the strategies for monitoring multipliers we saw in Chapter 5, in particular see 5.3.1, we can detect generic codimension one bifurcations of periodic orbits. Then, in principle we can continue curves of such points in two parameters, and detect higher codimension bifurcations as one does for branches of equilibria.

In the following example we see how to set up defining equations for curves of folds, flips, and Neimark-Sacker points, for the Poincarè map formulation. It will be left as an exercise to set up defining equations for the formulation as a boundary value problem.

EXAMPLE 1.1. Let $\boldsymbol{\phi}^t(\boldsymbol{u},\boldsymbol{\alpha})$ be the solution of the initial value problem

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\alpha}) \ , \ \ \boldsymbol{x}(0) = \boldsymbol{u} \,.$$

Here, $\boldsymbol{\alpha} \in \mathbb{R}^2$. A periodic orbit of period τ is given by the relations

$$\boldsymbol{\phi}^\tau(\boldsymbol{u},\boldsymbol{\alpha}) = \boldsymbol{u} \ , \ \ \sigma(\boldsymbol{u}) = 0 \ , \tag{1.10}$$

[1] First, look on ... at the files `README2` and `README3`. Once you fully understand them, proceed with the problem.

where $\sigma(\boldsymbol{u}) = (\boldsymbol{u}-\boldsymbol{u}^0)^T\boldsymbol{f}(\boldsymbol{u}^0)$. We rewrite (1.10) as

$$\boldsymbol{G}(\boldsymbol{u},\tau,\boldsymbol{\alpha}) = 0 \tag{1.11}$$

and consider its Jacobian at the solution $\boldsymbol{\phi}^\tau(\boldsymbol{u},\boldsymbol{\alpha})$

$$\boldsymbol{J} \;=\; \begin{bmatrix} \boldsymbol{\Phi}(\tau)-\boldsymbol{I} & \boldsymbol{f}(\boldsymbol{u}) & \partial_{\boldsymbol{\alpha}}\boldsymbol{\phi}^t(\boldsymbol{u},\boldsymbol{\alpha}) \\ D\sigma(\boldsymbol{u}) & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}. \tag{1.12}$$

$\boldsymbol{J}$ is of rank $n+1$ at generic codimension 1 bifurcations. In particular, there, the matrix

$$\begin{bmatrix} \boldsymbol{G}_{\boldsymbol{u}} & \boldsymbol{G}_\tau \end{bmatrix} \;=\; \begin{bmatrix} \boldsymbol{\Phi}(\tau)-\boldsymbol{I} & \boldsymbol{f}(\boldsymbol{u}) \\ D\sigma(\boldsymbol{u}) & \boldsymbol{0} \end{bmatrix} \tag{1.13}$$

will have rank $n+1$ or n or $n-1$. The eigenvalues of this matrix can be used for setting up continuation of generic codimension one bifurcations.

Now, let us recall that $1 \in \sigma(\boldsymbol{\Phi}(\tau))$ because of periodicity. As we know, in general, for an isolated periodic orbit 1 is a simple eigenvalue of $\boldsymbol{\Phi}(\tau)$, and since $D\sigma(\boldsymbol{u})\boldsymbol{f}(\boldsymbol{u}) \neq 0$, it follows (recall Theorem 4.2.1) that $\operatorname{rank}([\boldsymbol{G}_{\boldsymbol{u}}\ \boldsymbol{G}_\tau]) = n+1$ if 1 is a simple eigenvalue.

- *Fold.* At a fold, 1 is a double eigenvalue.

  PROBLEM 1.4. Show that, at fold, $\operatorname{rank}([\boldsymbol{G}_{\boldsymbol{u}}\ \boldsymbol{G}_\tau]) \leq n$. (Hint: see the proof of Theorem 4.2.1.)

  From the previous problem, in particular, it follows that $\det([\boldsymbol{G}_{\boldsymbol{u}}\ \boldsymbol{G}_\tau]) = 0$ at a fold, and hence along a curve of folds. To sum it up, to follow a curve of folds, we can augment the system (1.11) with the condition (cfr. with (6.4.5) and (6.4.6))

$$\det([\boldsymbol{G}_{\boldsymbol{u}}\ \boldsymbol{G}_\tau]) = 0\,, \qquad \text{or} \qquad \{1,1\} \in \sigma(\boldsymbol{\Phi}(\tau))\,. \tag{1.14}$$

- *Flip.* In this case, cfr. with (6.4.5) and (6.4.6), we can impose the condition

$$-1 \in \sigma(\boldsymbol{\Phi}(\tau))\,. \tag{1.15}$$

- *Neimark-Sacker.* In this case, cfr. with (6.4.7), we can impose the condition

$$e^{\pm i\phi} \in \sigma(\boldsymbol{\Phi}(\tau))\,. \tag{1.16}$$

Now, whichever branch we are following (i.e., folds, flips, or Neimark-Sacker points), we have an underdetermined system from $\mathbb{R}^{n+3} \to \mathbb{R}^{n+2}$. One may use a Gauss-Newton method on this system, or – as we saw in Chapter 6– one may parametrize the branch we are following in terms of pseudo arc-length, which would result in an extra condition of the type

$$\eta(\boldsymbol{u}(s),\tau(s),\boldsymbol{\alpha}(s)) \;=\; 0\,.$$

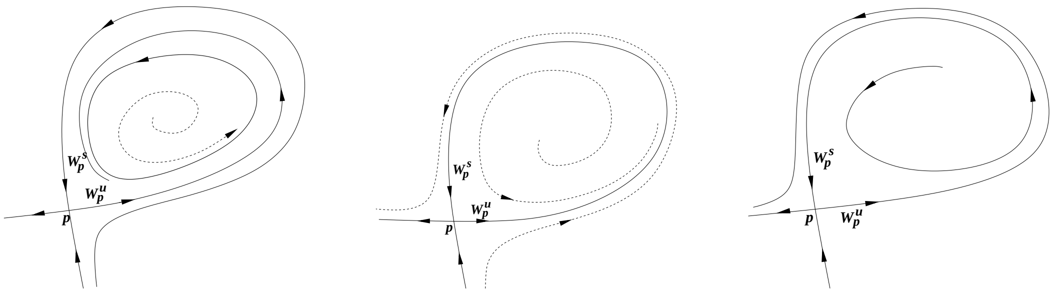
PROBLEM 1.5. Setup continuation equations for folds, flips, and Neimark-Sacker points of periodic orbits, by using the formulation as a boundary value problem (see (1.7)).

Other bifurcations for periodic solutions. In addition to the bifurcations of the Poincarè map there are also bifurcations which come from the loss of the definability of that map. The most important ones are

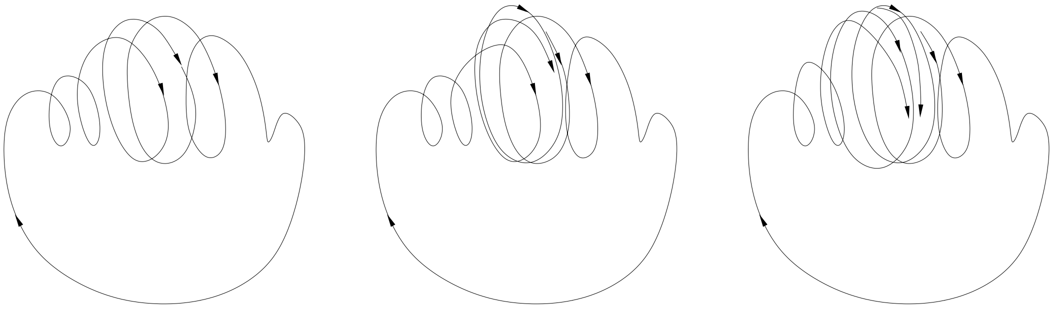
A The length of the orbit shrinks to 0. Usually this is just "reverse Hopf bifurcation": the periodic orbit terminates in a Hopf point on a branch of equilibria.

B The period approaches $+\infty$.

  i) The orbit remains of finite length, and the speed along the orbit decreases to 0 at some point. In this case, one can continue of the branch of periodic orbits as usual, trying to increase the period up to a reasonably large value. We will shortly see other techniques for this particular bifurcation in the context of *homoclininc* orbits.



  ii) The length of the orbit, as well as the period, approaches $+\infty$, while the orbit itself stays bounded, and away from equilibria. This type of bifurcation is called a *blue sky catastrophe*. It can happen as follows: a stable periodic orbit Γ_α becomes homoclinic to a non-hyperbolic periodic orbit H, which eventually splits (fold of periodic orbits) in two hyperbolic orbits H_1 and H_2 after which point no periodic orbit close to Γ_α exist.



2. Connecting orbits

Motivated by case B i) above, we discuss now homoclinic orbits. The first step is to realize that in case B i) there must be a fixed point along the orbit.

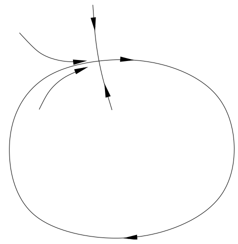
A *homoclinic* orbit $\boldsymbol{\phi}^t(\boldsymbol{u})$ to an equilibrium $\boldsymbol{p}$ is one that satisfies

$$\lim_{t\to-\infty} \boldsymbol{\phi}^t(\boldsymbol{u}) = \lim_{t\to+\infty} \boldsymbol{\phi}^t(\boldsymbol{u}) = \boldsymbol{p} \ .$$

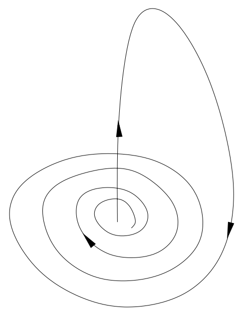
Clearly a homoclinic orbit lies in the intersection $W^s_{\boldsymbol{p}} \cap W^u_{\boldsymbol{p}}$.

Above in case B i) we saw a homoclinic orbit of a hyperbolic equilibrium.

EXAMPLE 2.1. On the right we have a figure of a homoclinic orbit at a saddle node bifurcation. Here the equilibrium has eigenvalue zero and the center manifold is topologically a circle.



EXAMPLE 2.2. In higher dimensions, there are different ways in which we can approach $\boldsymbol{p}$, depending on the eigenvalues of $\boldsymbol{f_x}(\boldsymbol{p})$. On the right the equilibrium is hyperbolic and the stable eigenvalues are complex. This is called a *Silnikov* bifurcation.



Now we will consider the case of $\boldsymbol{p}$ being a hyperbolic equilibrium. Then $W^{s,u}_{\boldsymbol{p}}$ are unique and perturb smoothly. This is a consequence of center manifold theorem 6.3.1. Suppose we have a homoclinic orbit to an equilibrium $\boldsymbol{p}$ of the system $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \alpha_0)$. As we saw in case (b)(i) we can perturb the homoclinic orbit away. This is true for flows. The case of maps is very different as we will see later.

A homoclinic orbit means a nontrivial intersection of the manifolds $W^{s,u}_{\boldsymbol{p}}$. If we take two manifolds "in general position" which intersect along a set and we perturb them, then we should expect the perturbed manifolds to intersect along a set close to the original one. For example, think of two curves in a plane which intersect at a point $\boldsymbol{p}$, and perturb the curves smoothly. Intuitively, we expect that there is an intersection point close to $\boldsymbol{p}$. This is guaranteed if the original curves had different directions at $\boldsymbol{p}$. For general cases we need the concept of *transversality.*

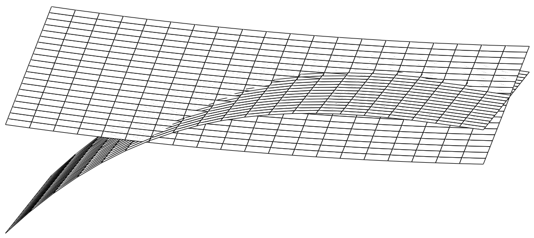
Given two submanifolds M and N of $\mathbb{R}^n$. Let $\boldsymbol{x} \in (M \cap N)$, and let $T_{\boldsymbol{x}}M$ and $T_{\boldsymbol{x}}N$ be the tangent spaces of M and N at $\boldsymbol{x}$. We say that M and N *are transversal* *at* $\boldsymbol{x}$, if their tangent spaces span $\mathbb{R}^n$:

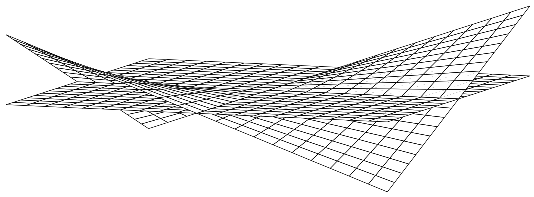
$$T_{\boldsymbol{x}}M \ + \ T_{\boldsymbol{x}}N = \mathbb{R}^n \ . \tag{2.1}$$

This is denoted by $M \pitchfork_{\boldsymbol{x}} N$. Further, we say that M and N *are transversal* (denoted $M \pitchfork N$) if $M \pitchfork_{\boldsymbol{x}} N$ for all $\boldsymbol{x} \in M \cap N$.

The following are examples of transversal manifolds

- two curves in $\mathbb{R}^2$, intersecting at nonzero angle;
- two nonintersecting curves in $\mathbb{R}^3$;
- a curve intersecting a surface in $\mathbb{R}^3$ at nonzero angle;
- two surfaces in $\mathbb{R}^3$ intersecting along a curve at nonzero angle.


Transversal


Not transversal (at one point)

The following results as well as other consequences of transversality can be found in [**Hir76**].

LEMMA 2.1. *If* M *and* N *are transversal, then their intersection (if not empty) is a submanifold of dimension* $\dim(M)+\dim(N)-n$. *Further, if* $M \cap N$ *is compact then small* $\mathcal{C}^1$ *–perturbations of* M *and* N *will also be transversal.*

Usually nontransversal intersections change under perturbations.

PROBLEM 2.1. Perturb the nontransversal intersection in the figure above to a transversal one.

A trajectory $\gamma = \left\{ \boldsymbol{\phi}^t(\boldsymbol{u}) \,\middle|\, t \in \mathbb{R} \right\}$ satisfying

$$\lim_{t \to -\infty} \boldsymbol{\phi}^t(\boldsymbol{u}) \ = \ \boldsymbol{p}, \qquad \text{and} \qquad \lim_{t \to +\infty} \boldsymbol{\phi}^t(\boldsymbol{u}) \ = \ \boldsymbol{q} .$$

is called a *connecting orbit* between the equilibria $\boldsymbol{p}$ and $\boldsymbol{q}$. If $\boldsymbol{p} \neq \boldsymbol{q}$, then the orbit is called heteroclinic, otherwise homoclinic.

Let γ be such an orbit. Clearly it lies in the intersection of $W^u_{\boldsymbol{p}}$ and $W^s_{\boldsymbol{q}}$. Let $n^u_{\boldsymbol{p}}$, $n^s_{\boldsymbol{q}}$ denote the dimensions of these. If $W^u_{\boldsymbol{p}} \pitchfork W^s_{\boldsymbol{q}}$, then the intersection has dimension ≥ 1 and

$$n^u_{\boldsymbol{p}} + n^s_{\boldsymbol{q}} \geq n + 1 \ .$$

Consider the case in which this is not satisfied but we have a system depending on p parameters:

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\alpha}) \ .$$

Let $\boldsymbol{p}, \boldsymbol{q}$ be hyperbolic equilibria for $\boldsymbol{\alpha}$ in a p-dimensional neighborhood of $\boldsymbol{\alpha}_0$. Consider the manifolds (foliated by $\boldsymbol{\alpha}$)

$$(2.2) \qquad \widetilde{W}_0^u = \{(W_0^u(\boldsymbol{\alpha}), \boldsymbol{\alpha}) \ : \ \boldsymbol{\alpha} \in U(\boldsymbol{\alpha}_0)\} \ , \quad \widetilde{W}_1^s = \{(W_1^s(\boldsymbol{\alpha}), \boldsymbol{\alpha}) \ : \ \boldsymbol{\alpha} \in U(\boldsymbol{\alpha}_0)\}$$

in $\mathbb{R}^{n+p}$. They have dimensions $n_{\boldsymbol{p}}^u + p$ and $n_{\boldsymbol{q}}^s + p$, respectively. Thus a transversal nonempty intersection can exist only if

$$n_{\boldsymbol{p}}^u + p + n_{\boldsymbol{q}}^s + p \geq n + p + 1 \ , \qquad \text{i.e.,} \qquad p \geq n + 1 - n_{\boldsymbol{p}}^u - n_{\boldsymbol{q}}^s \ .$$

Especially for a homoclinic trajectory we need $p \geq 1$.

For setting up a system of equations defining a connecting orbit note first that if $\boldsymbol{x}(t, \boldsymbol{\alpha})$ represents a connecting orbit, then $\boldsymbol{x}(t + s, \boldsymbol{\alpha})$ represents the same orbit, for any $s \in \mathbb{R}$; in other words, we need to fix the time shift to characterize the orbit in a unique way. Here the *phase condition* can be any of the choices discussed in Chapter 5.3.2. That is,

$$(2.3) \qquad \begin{aligned}
&\text{(a)} \qquad \sigma(\boldsymbol{x}) = (\boldsymbol{x}(0) - \boldsymbol{x}^0(0))^T \ (\boldsymbol{x}^0)'(0) \ , \\
&\text{(b)} \qquad \sigma(\boldsymbol{x}) = \int_0^1 \ (\boldsymbol{x}(t) - \boldsymbol{x}^0(t))^T \ \boldsymbol{x}'(t) dt \ , \\
&\text{(c)} \qquad \sigma(\boldsymbol{x}) = \int_0^1 (\boldsymbol{x}'(t) - (\boldsymbol{x}^0)'(t))^T \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x}(t)) \boldsymbol{x}'(t) dt \ .
\end{aligned}$$

Thus we can characterize a connecing orbit as solution of the problem:

$$(2.4) \qquad \begin{cases}
\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{\alpha}) \\
\boldsymbol{f}(\boldsymbol{p}, \boldsymbol{\alpha}) = 0 \ , \quad \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{\alpha}) = 0 \\
\lim\limits_{t \to -\infty} \boldsymbol{x}(t) = \boldsymbol{p} \ , \quad \lim\limits_{t \to \infty} \boldsymbol{x}(t) = \boldsymbol{q} \\
\sigma(\boldsymbol{x}) = 0 \, .
\end{cases}$$

Next we will study whether this problem is *well posed*: does a nearby problem also have a unique solution? We already discussed this issue for boundary value problems on a finite interval (see Section 4.2). Things here are more complicated because the problem is on the entire real line and the boundary conditions are in essence a statement on the need to restrict attention for possible solutions to those functions which converge at $\pm\infty$; thus, a heavier machinery is needed. However, the fundamental idea is the same as in Chapter 4: with reasonable asumptions the solution will represent a regular point of a map.

We delay all technical aspects of this discussion to the appendix, and here merely state the result:

THEOREM 2.2. *Let* $p = n + 1 - n^u_{\boldsymbol{p}} - n^s_{\boldsymbol{q}}$, $\widetilde{W}^u_{\boldsymbol{p}} \pitchfork \widetilde{W}^s_{\boldsymbol{q}}$ *with nonempty intersection, and* $\sigma_{\boldsymbol{x}}\boldsymbol{x}' \neq 0$. *Then* (2.4) *is well posed.*

We are ready to address the following questions.

- When should we expect a problem to have a connecting orbit?
- How should one solve (2.4) numerically?

Let us begin to address the first question by looking at some applications/examples where connecting orbits arise.

2.1. Connecting orbits in applications. Connecting orbits are ancountered in many situations, and the references at the end of this chapter provide many applications. The following situations are typical.

Homoclinic orbits are seen often at bifurcations of periodic orbits (as the period goes to ∞). An instance of this is Example 2.6 below. People have been interested in heteroclinic orbits especially in the context of reaction diffusion and other so called gradient systems, e.g., see [**BC84**] and [**BSS93**]. These systems have well understood (and rather limited) dynamics: fixed points and their connections, the heteroclinic orbits.

An interesting and rich class of problems which give rise to connecting orbits are parabolic partial differential equations via the so-called traveling wave ansatz.

EXAMPLE 2.3. [Traveling wave] Traveling waves are the simplest instance of a nontrivial (i.e., not constant) solution of a parabolic PDE. Consider the following parabolic equation

$$u_t = u_{xx} + g(u, \alpha)\ , \quad t > 0\ , \quad -\infty < x < +\infty\ . \tag{2.5}$$

A *traveling wave* of (2.5) is a special solution of the form $u(x,t) = v(x + ct)$, where c is termed the *wave speed*. If we take $z = x + ct$ as new independent variable, we get an ordinary differential equation:

$$v'' - cv' + g(v, \alpha) = 0\ , \quad -\infty < z < +\infty\ . \tag{2.6}$$

Suppose that v_0 and v_1 are roots of g. A *front wave* of (2.5) is a non-constant solution of (2.6) such that $\lim_{z\to\pm\infty} v = v_{0,1}$, respectively. In other words, the front wave is given by solutions of the following boundary value problem on the real line

$$\begin{cases} v'' - cv' + g(v, \alpha) = 0 \\ \lim\limits_{z\to-\infty} v(z, \alpha) = v_0\ , \quad \lim\limits_{z\to\infty} v(z, \alpha) = v_1\ . \end{cases} \tag{2.7}$$

Thus, quite clearly, if $v_0 = v_1$ the front wave is a homoclinic connection (usually called a *pulse wave*), while if $v_0 \neq v_1$ the front wave is an heteroclinic connection. In the next example, we will study the latter case in detail on a prototypical problem. Later, in Example 2.5 and Problem 2.5, we will see an instance of a pulse wave.

EXAMPLE 2.4. [Nagumo equation] This equation is a simplification of the Fitz-Hugh-Nagumo equations, which we will meet in their complete form in Example 2.5. In (2.5), one has

$$g(u,\alpha) = u(1-u)(u-\alpha)\,, \quad 0<\alpha<1\,.$$

We rewrite (2.6) as the 1st order system

$$\frac{d}{dz}\begin{bmatrix} v \\ v' \end{bmatrix} = \begin{bmatrix} v' \\ cv' - v(1-v)(v-\alpha)' \end{bmatrix} \qquad \text{or} \qquad \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x},\alpha),\ \boldsymbol{x} = \begin{bmatrix} v \\ v' \end{bmatrix}\,.$$

There are three equilibria: $\boldsymbol{p} = (0,0)^T,\ \boldsymbol{q} = (1,0)^T,\ \boldsymbol{r} = (\alpha,0)^T$. The Jacobian is $D\boldsymbol{f} = \begin{bmatrix} 0 & 1 \\ 3v^2 - 2v(1+\alpha)+\alpha & c \end{bmatrix}$, and at the three equilibria we have

$$\sigma(D\boldsymbol{f}(\boldsymbol{p})) = \{\frac{1}{2}(c \pm \sqrt{c^2+4\alpha})\}\,,\ \sigma(D\boldsymbol{f}(\boldsymbol{q})) = \{\frac{1}{2}(c \pm \sqrt{c^2+4(1-\alpha)})\}\,,$$
$$\text{and} \quad \sigma(D\boldsymbol{f}(\boldsymbol{r})) = \{\frac{1}{2}(c \pm \sqrt{c^2-4\alpha(1-\alpha)})\}\,.$$

Observe that $\boldsymbol{p}$ and $\boldsymbol{q}$ are saddles for all $\alpha \in (0,1)$, while the nature of $\sigma(D\boldsymbol{f}(\boldsymbol{r}))$, depends on c. Since $\boldsymbol{p}$ and $\boldsymbol{q}$ are hyperbolic, we suspect that there may be a heteroclinic connection between them. Notice that we have $n^u_{\boldsymbol{p}} = n^s_{\boldsymbol{q}} = 1$, $n=2$, and one free parameter, α. To find the heteroclinic connection, we look for solutions of

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x},\alpha)\,, \quad -\infty < z < \infty\,, \\ \lim_{z\to-\infty} \boldsymbol{x}(z,\alpha) = \boldsymbol{p}\,, \ \lim_{z\to\infty} \boldsymbol{x}(z,\alpha) = \boldsymbol{q}\,. \end{cases}$$

Surprisingly enough, this problem can be solved analytically. Given the type of decay we want at $\pm\infty$, we look for a solution of the form

$$\boldsymbol{x} = \begin{bmatrix} v \\ v' \end{bmatrix}\,, \qquad \text{with} \qquad v = \frac{e^{az}}{A+e^{az}}\,, \quad A>0\,.$$

Plugging this candidate solution into the differential equation gives the requirement

$$A^2e^{az}(a^2-ac-\alpha) + Ae^{2az}(1-a^2-ac-\alpha) = 0\,,$$

from which we must have

$$a^2 - ac - \alpha = 0\,, \qquad \text{and} \qquad 1-a^2-ac-\alpha = 0\,.$$

Solving this, we get $a = \pm\frac{1}{\sqrt{2}}$ and $c = \pm\frac{1-2\alpha}{\sqrt{2}}$. Therefore, we have found two solutions:

$$\boldsymbol{x}(z,\alpha) = \begin{bmatrix} v \\ v' \end{bmatrix} \ , \ z = x + ct \ , \quad \text{and}$$

$$(a)\ v = \frac{e^{z/\sqrt{2}}}{A + e^{z/\sqrt{2}}} \ , \quad c = \frac{1-2\alpha}{\sqrt{2}} \ , \ A > 0 \ ,$$

$$(b)\ v = \frac{e^{-z/\sqrt{2}}}{A + e^{-z/\sqrt{2}}} \ , \quad c = -\frac{1-2\alpha}{\sqrt{2}} \ , \ A > 0 \, .$$

The first connects $\boldsymbol{p}$ (approached as $z \to -\infty$) to $\boldsymbol{q}$ (as $z \to \infty$), the second connects $\boldsymbol{q}$ (as $z \to -\infty$) to $\boldsymbol{p}$ (as $z \to \infty$). What we have found is an *heteroclinic cycle*; it looks like an oval. Observe that the positive constant A is undetermined. The undeterminacy of A is a reflection of the time shift ambiguity inherent in the characterization of the connecting orbit (i.e., of the traveling wave) as solution of (2.7).

PROBLEM 2.2. Convince yourself of the last statement; i.e., "The undeterminacy ... ". (Hint: take a solution $v = \frac{e^{z/\sqrt{2}}}{A+e^{z/\sqrt{2}}}$ and another $w = \frac{e^{(z+\tau)/\sqrt{2}}}{B+e^{(z+\tau)/\sqrt{2}}}$ with $A \neq B$, and verify that you can find τ).

Thus, to fix A, we must impose a phase condition. (A simple choice in this case could be to force $v = 1/2$ at $z = 0$, which gives $A = 1$). Finally, observe that we have obtained a branch of traveling waves solution of (2.5) for $\alpha \in (0,1)$. All traveling waves look identical in phase space, and we have an heteroclinic cycle; the behavior of trajectories in the region of phase space within the heteroclinic cycle depends on the value of α, and on $\sigma(D\boldsymbol{f}(\boldsymbol{r}))$. An interesting case to consider is $\alpha = 1/2$, in which case both eigenvalues of $\sigma(D\boldsymbol{f}(\boldsymbol{r}))$ are purely imaginary: the phase space within the heteroclinic cycle is "filled" with periodic orbits.

PROBLEM 2.3. Complete the study of local dynamics in phase space for this problem by studying how points inside the heteroclinic cycle evolve as α varies in $(0,1)$. May you draw a complete pictures of the dynamics in phase space? (Hint: Inside the heteroclinic cycle, there are several local bifurcations and qualitatively different behaviors as $\alpha \in (0, 1/2-1/\sqrt{6}), (1/2-1/\sqrt{6}, 1/2), (1/2, 1/2+1/\sqrt{6}), (1/2+1/\sqrt{6}, 1))$.

PROBLEM 2.4. Consider the traveling wave ODE for the Nagumo PdE, that is

$$v'' - cv' + v(1-v)(v-\alpha) = 0 \, , \quad 0 < \alpha < 1 \, .$$

Rewrite it as a first order system in $\begin{bmatrix} v \\ v' \end{bmatrix}$. To find the heteroclinic connection between $(0,0)$ and $(1,0)$, we explore the idea that the connection itself is an invariant manifold for the flow of the system. To do so, we demand that v' is a function of v: $v' = \phi(v)$, and use this relation in the differential equation. We end up with the system (do the calculations)

$$\phi_v \phi = c\phi - v(1-v)(v-\alpha)$$
$$\phi(0) = 0\,, \quad \phi(1) = 0\,.$$

Notice that there are two boundary conditions, but just one equation.

(i) Solve this differential equation and find out what the graph $\phi(v)$ is. You should end up with a solution only for restricted values of c.

(ii) Next, consider the variational equation for the ϕ-equation around the solution you just found; that is, let $\phi \to \phi + \psi$ and end up with a linear equation for ψ. Discuss the solution of this variational equation and explain what is going on.

(iii) When and how do you think the way of proceeding laid down in this Problem may lend itself to a numerical procedure?

EXAMPLE 2.5. [FitzHugh-Nagumo equation] Pulse waves (i.e., homoclinic) can also occur as traveling waves of coupled partial differential equations. They are often called *solitary waves*. For example, they occur in the FitzHugh-Nagumo model

$$\begin{aligned} u_t &= u_{xx} + u - \frac{1}{3}u^3 - r \\ r_t &= d(u + a - br)\,. \end{aligned} \tag{2.8}$$

Here, a, b, d are constants, and the problem is posed on the half plane: $t \geq 0$, $-\infty < x < \infty$. The interested reader can find some of the history behind this model, as well as numerical computations for the solitary wave, in [**Miu82**] and references therein.

Now, there is an equilibrium of (2.8), namely (not easy to find)

$$u_0 = br_0 - a\,, \; r_0 = \frac{1}{b}\big[a - 4\frac{1-b}{b}\sinh y\big]\,, \; y = \frac{1}{3}\sinh^{-1}\big[\frac{3a}{2b}(\frac{b}{1-b})^{3/2}\big]\,.$$

Changing variables: $u \leftarrow u - u_0\,, \;\; r \leftarrow r - r_0$, brings the equilibrium at the origin, and (2.8) rewrites

$$\begin{aligned} u_t &= u_{xx} + (1-u_0^2)u - \frac{1}{3}u^3 - r \\ r_t &= d(u + a - br)\,. \end{aligned}$$

From this, after the traveling wave ansatz $z = x + ct$, in the new variables $v = u(z)$, $\rho = r(z)$, we get the system of ODEs

$$\begin{cases} v'' - cv' + (1-u_0^2)v - u_0 v^2 - \dfrac{1}{3}v^3 - \rho = 0 \\ c\rho' - d(v - b\rho) = 0\,. \end{cases} \tag{2.9}$$