Teknillinen korkeakoulu
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Exercise 9. (5.4.)

1. a) Let $g \in S_n$ and let $c = (a_1 \ \dots \ a_p) \in S_n$ be a p-cycle. Show that gcg^{-1} is a p-cycle $(g(a_1) \ \dots \ g(a_p))$.
   b) Show that all cycles of the same length are conjugate in S_n.
   c) Show that two permutations $a, \ b \in S_n$ are in the same conjugacy class if and only if there is an equal number of cycles of given length in their representations as products of disjoint cycles.

2. a) Find the conjugacy classes of the elements in S_3.
   Ans. Three classes.
   b) Find the conjugacy classes of the elements in S_4.
   Ans. Five classes.

3. a) Using the Wirtinger relations, show that all generators in the group G of a PL knot are conjugate to each other; i.e. if $a, \ b$ are generators, then $a = gbg^{-1}$ with a suitable $g \in G$.
   b) A presentation for the group of the knot 4_1 is

   $$G = < a, b \mid a^{-1}bab^{-1}ab = ba^{-1}ba > .$$

   Show that the relation can be written as $a = (b^{-1}aba^{-1})b(b^{-1}aba^{-1})^{-1}$.
   c) Show that there is no surjective homomorphism $\varphi \colon G \to S_3$.
   Hint: The generators will be mapped into the same conjugacy class (why?), and this reduces the number of cases needed.
   d) Show that the group G of 4_1 is not abelian by using a suitable homomorphism $\varphi \colon G \to S_4$.
   Hint: Choose e.g. $\varphi(a) = (1 \ 2 \ 3 \ 4)$.

4. Consider the $2n$-component link K. Its group G is generated by $x_i, \ y_i, \ z_i, \ w_i$, $0 \le i \le n-1$, as in the figure.
   a) Find the four relations in neighbouring crossings.
   b) Show that there is a surjective homomorphism
   $\varphi \colon G \to F_2 = < \alpha, \beta \mid - >$, such that
   $\varphi(x_i) = \alpha, \ \varphi(y_i) = \beta, \ \varphi(z_i) = \beta^{-1}\alpha\beta, \ \varphi(w_i) = \alpha^{-1}\beta\alpha$ for all i.
   c) Write the loop γ in terms of the generators and show, using b), that γ is not trivial in the complement of K.

Kääntöpuolella on pääsiäisloman ratoksti oheislukemista torus-solmuista, joista yksinkertaisin on kolmiapila 3_1.