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Exercise 7. (15.3.)

1. Show that $< a, b \mid (ab)^2ab^2 = 1 > \cong < \alpha \mid - > \cong \mathbf{Z}$.
   Hint: Let first $x = ab,\ y = ab^2$ and eliminate $a,\ b$. You may proceed without giving all the formal Tietze transformations.

2. For the symmetric group S_3 we have derived presentations
   $$< a, b \mid a^2 = 1, b^3 = 1, ba = ab^2 > \quad \text{and} \quad < \alpha, \beta \mid \alpha^2 = 1, \beta^3 = 1, (\alpha\beta)^3 = 1 > .$$
   Show that these groups are really isomorphic by giving the required Tietze transformations.
   Hint: Previously, we had $a = (1\ 3),\ b = (1\ 2\ 3),\ \alpha = (1\ 2)$ and $\beta = (1\ 3)$. Using these, find a suitable substitution.

3. Show that
   $$< a, b \mid a^4 = 1, b^4 = 1, ba = a^2b^2 > \cong < \alpha, \beta \mid \alpha^5 = 1, \beta^4 = 1, \alpha\beta = \beta\alpha^2 >,$$
   when we substitute $\alpha = ba,\ \beta = b$.
   Hint: First justify the following formulas:
   $$(ba)^2 = baba = a^2b^2ba = a^2b^{-1}a = a^3a^{-1}b^{-1}a = a^3b^2a^3 = aa^2b^2a^3 = aba^4 = ab,$$
   so that
   $$(ba)^6 = (ab)^3 = a(ba)^2b = a^2b^2 = ba.$$

4. a) Calculate the number of elements of the group in the preceding exercise. Consider two different possibilities:
   (i) First presentation: The last relation may lead to the idea, that the normal forms could be chosen to be a^mb^n, so there would be 16 elements. This is not true; why? (Try to transform $baba$ to the form above, for example).
   (ii) Second presentation: Show that the words $\beta^m\alpha^n$ can be chosen as normal forms and solve the problem using this. (A reasonable explanation without a careful proof is enough.)
   b) How can you get the normal forms for the first presentation using Problem 3 and part (ii) above?