Teknillinen korkeakoulu
Johdatus solmuteoriaan, kevät 2002
Exercise 5. (1.3.)

Solve four out of the following five problems.

1. (VäII, 21.9) Show that a continuous map $f\colon X \to Y$ is a homotopy equivalence $\Leftrightarrow$ there are continuous $g_1,\ g_2\colon Y \to X$ such that $g_1 \circ f \simeq \mathrm{id}_X$ and $f \circ g_2 \simeq \mathrm{id}_Y$.
   Hint: Consider the map $g_1 \circ f \circ g_2$.

2. (VäII, 22.2) Show that in a pathwise connected space, all paths are homotopic to each other. (Here it is not required that the end points are kept fixed)

3. (VäII, 23.4) Let X and Y be patwise connected sets and let $p\colon X \times Y \to X$, $q\colon X \times Y \to Y$ be the projections $p(x,y) = x$, $q(x,y) = y$. Show that the mapping
   $f\colon \pi(X \times Y, (x_0, y_0)) \to \pi(X, x_0) \times \pi(Y, y_0)$, $f(\bar{\alpha}) = (p_*(\bar{\alpha}), q_*(\bar{\alpha}))$,
   is a group isomorphism.
   Note: On the right hand side the group operation is of the form $(a,b)(a',b') = (aa', bb')$.

4. (VäII, 23.6) Show that a space X is simply connected $\Leftrightarrow$ all continuous mappings $f\colon S^1 \to X$ are homotopic to each other.
   Hint: Problem 2.

5. Prove the so called Lebesgue covering theorem [VäI, 13.33]: If $A \subset \mathbf{R}^n$ is a compact set and $A \subset \cup_{i\in I} D_i$, where the sets D_i are open, then there is a real number $\lambda > 0$ such that: if $x \in A$, then $B(x, \lambda) \subset D_i$ for some $i \in I$.
   Hint: Suppose this is not true: then none of the numbers $\lambda = 1/n$, $n \in \mathbf{N}$, satisfies the condition, and for each n one can choose a point x_n, which breaks it. Consider the sequence (x_n) and use (sequential) compactness.