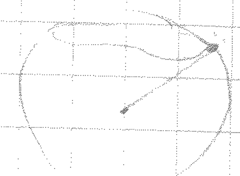
$\sigma_u = v\gamma'$ $\sigma_v = \gamma$



$E = \|\sigma_u\|^2 = v^2\|\gamma'\|^2 = v^2$

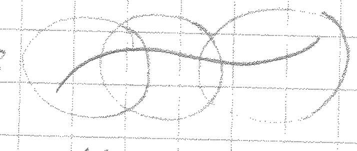
$F = \sigma_u \cdot \sigma_v = v\gamma' \cdot \gamma = 0$

$G = \|\sigma_v\|^2 = \|\gamma\|^2 = 1$

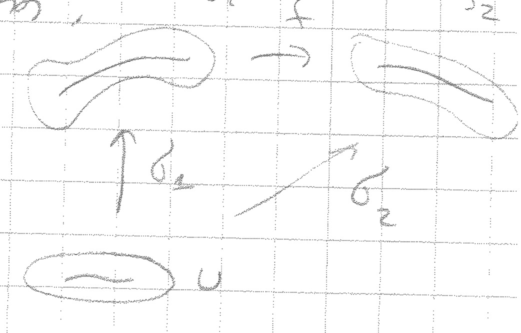
$\Rightarrow$ 1. perusmuoto $v^2 du^2 + dv^2$ riippumaton polusta γ!

<u>Määritelmä</u> Ol. S_1, S_2 sään. pintoja. Diffeomorfismi $f: S_1 \to S_2$ on <u>isometria</u> jos se kuvaa S_1:n polut samanpituisiksi poluiksi pinnalla S_2.

2.1. <u>Lause</u> Diffeomorfismi $f: S_1 \to S_2$ on isometria jos ja vain jos S_1:n sään. parametrisoinnilla σ_1 ja S_2:n parametrisoinnilla $f \circ \sigma_1$ on sama ensimmäinen perusmuoto.

<u>Tod</u>: Riittää tarkastella yhtä säännöllistä parametrisointia $\sigma_1: U \to \mathbb{R}^3$ (additiivisuus ) L.1.2. $\Rightarrow$

$\sigma_2 = f \circ \sigma_1$ S_2:n sään. parametrisointi.



"$\Leftarrow$" Ol. $t \mapsto (u(t), v(t))$ U:n sään. polku

$\gamma_1: \gamma_1(t) = \sigma_1(u(t), v(t)) \in S_1$

$\gamma_2: \gamma_2(t) = \sigma_2(u(t), v(t)) \in S_2$

Tällöin $f(\gamma_1(t)) = f(\sigma_1(u(t), v(t))) = \sigma_2(u(t), v(t)) = \gamma_2(t)$ $\forall t$

Molemmilla poluilla sama pituus, sillä se saadaan integroimalla lauseketta $(Eu'^2 + 2Fu'v' + Gv'^2)^{1/2}$, missä $E\,du^2 + 2F\,du\,dv + G\,dv^2$ yleinen 1. perusmuoto.

"$\Rightarrow$" Ol. f isometria ja $t \mapsto (u(t), v(t))$ U:n polku. Poluilla $\gamma_1(t) = \sigma_1(u(t), v(t))$ ja $\gamma_2(t) = \sigma_2(u(t), v(t))$

on sama pituus. Ol. (α, β) parametriväli. $\forall\, t_0, t \in (\alpha, \beta)$

$$\int_{t_0}^{t} (E_1 u'^2 + 2F_1 u'v' + G_1 v'^2)^{1/2} dt = \int_{t_0}^{t} (E_2 u'^2 + 2F_2 u'v' + G_2 v'^2)^{1/2} dt,$$

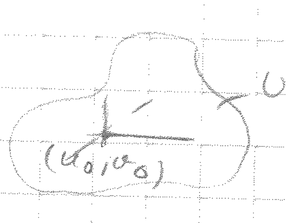
missä E_1, F_1, G_1 σ_1:n 1. perusmuodon kertoimet
E_2, F_2, G_2 σ_2:n — " — — " —

$$\overset{\forall t_0, t}{\Rightarrow} \quad E_1 u'^2 + 2F_1 u'v' + G_1 v'^2 = E_2 u'^2 + 2F_2 u'v' + G_2 v'^2 \qquad (*)$$

Olk. $t_0 \in (\alpha, \beta)$ ja $u_0 = u(t_0)$, $v_0 = v(t_0)$

Soveltamalla yhtälöä $(*)$ seuraaviin U:n polkuihin $t \mapsto (u(t), v(t))$

(i) $u(t) = u_0 + t - t_0$, $v \equiv v_0$ $\Rightarrow$ $u' \equiv 1$, $v' \equiv 0$



$\Rightarrow$ $E_1 = E_2$

(ii) $u \equiv u_0$, $v(t) = v_0 + t - t_0$ $\Rightarrow$ $u' \equiv 0$, $v' \equiv 1$

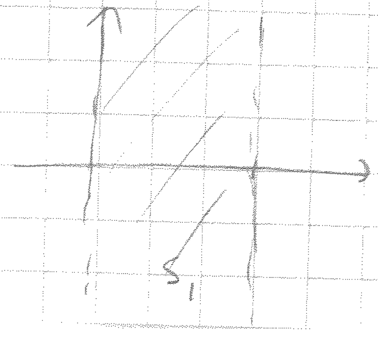
$\Rightarrow$ $G_1 = G_2$

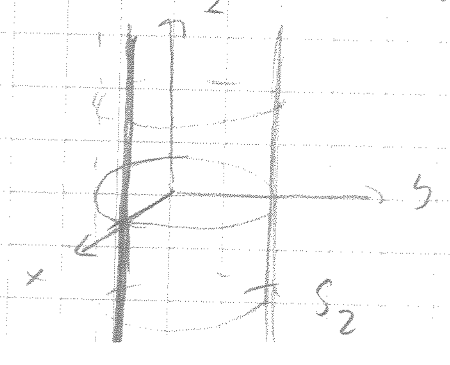
(iii) $u(t) = u_0 + t - t_0$, $v(t) = v_0 + t - t_0$ $\Rightarrow$ $u' \equiv 1 \equiv v'$

$$\Rightarrow \quad E_1 + 2F_1 + G_1 = E_2 + 2F_2 + G_2 \quad \underset{(ii)}{\overset{(i)}{\Rightarrow}} \quad F_1 = F_2 \quad . \square .$$

<u>Esim</u> $S_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid 0 < x < 2\pi\}$

$S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \setminus \{(1, 0, z) \mid z \in \mathbb{R}\}$





$\sigma_1 : (0,2\pi)\times\mathbb{R} \to \mathbb{R}^3 \qquad \sigma_1(u,v) = (u,v,0)$

koko S_1:n parametrisointi

$\sigma_2 : (0,2\pi)\times\mathbb{R} \to \mathbb{R}^3 \qquad \sigma_2(u,v) = (\cos u, \sin u, v)$

koko S_2:n parametrisointi. Esim 1° 3° edellä

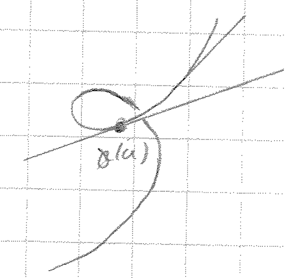
$\Rightarrow$ σ_1:n ja σ_2:n 1. perusmuoto sama $\Rightarrow$

$f: f(u,v,0) = (\cos u, \sin u, v) \qquad f: S_1 \to S_2$ on isometria.

<u>Esim</u> Käyrän tangentin kehittämä pinta $S = \bigcup_{u\in(\alpha,\beta)} \mathrm{sp}\{\gamma'(u)\}$.

$\gamma : (\alpha,\beta) \to \mathbb{R}^3$ sään. ja $\|\gamma'(t)\| = 1$

pisteen $p \in S$ ympäristön parametrisointi:

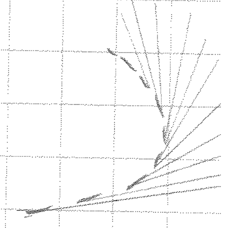


$$\sigma(u,v) = \gamma(u) + v\gamma'(u), \quad v \in \mathbb{R}$$

$$\sigma_u = \gamma'(u) + v\gamma''(u), \quad \sigma_v = \gamma'(u)$$

$\Rightarrow \sigma_u \times \sigma_v = v\gamma''(u)\times\gamma'(u)$ sään. $\forall v \neq 0 + \gamma''(u)$ ...

jos σ injektio.



Frenet: $\begin{cases} e_1 = \gamma' \\ \gamma'' = \kappa e_2 \\ e_3 = e_1 \times e_2 \end{cases} \rightarrow \sigma_u \times \sigma_v = v\kappa e_2 \times e_1 = -v\kappa e_3$

$\therefore$ σ sään $\Leftrightarrow$ σ inj, $\kappa \neq 0$, $v \neq 0$ (γ:n pisteet poistetaan)

2.2. <u>Lause</u> Käyrän tangentin kehittämä pinta on isometrinen tason avoimen osajoukon kanssa.

<u>Tod</u>: $E = \sigma_u\cdot\sigma_u = (\gamma' + v\gamma'')(\gamma' + v\gamma'') = \|\gamma'\|^2 + v^2\|\gamma''\|^2$
$= 1 + v^2\kappa^2$

kun $\|\gamma'\| = 1$

$F = \sigma_u \cdot \sigma_v = (\gamma' + v\gamma'') \cdot \gamma' = \|\gamma'\|^2 + v\gamma'' \cdot \gamma' = 1$

$G = \sigma_v \cdot \sigma_v = \gamma' \cdot \gamma' = 1$

$\Rightarrow$ 1. perusmuoto $ds^2 = (1 + v^2\kappa^2)\,du^2 + 2du\,dv + dv^2$

Pitää os: osa tason parametrisointanssa s.e. sama 1. perusmuoto

Olk $\tilde{\gamma}$ tasokäyrä s.e. $\tilde{\kappa}_s = \kappa > 0$. Tällainen $\tilde{\gamma}$

aina löytyy L.I.2.3. perusteella. $\tilde{\gamma}$:n tangentin kehittämän

pinnan parametrisointi $\tilde{\sigma}(u,v) = \tilde{\gamma}(u) + v\tilde{\gamma}'(u)$

$\Rightarrow \tilde{E} = E,\ \tilde{F} = F,\ \tilde{G} = G$ kun $\|\tilde{\gamma}'\| = 1$.

Lisäksi $|\tilde{\gamma}| \subset \mathbb{R}^2 \Rightarrow \tilde{\gamma}'(t) \in \mathbb{R}^2 \ \forall t$

$\Rightarrow \tilde{\sigma}(u,v) = \tilde{\gamma}(u) + v\tilde{\gamma}'(t)$ avoimen tasojoukon parametrisointi kun $v \neq 0$ ja $\tilde{\sigma}$ inj. □.

<u>Huom</u> Voidaan os. myös käänteinen väite: ∀ riittävän pieni pinnan osa, joka isometrinen tason avoimen joukon kanssa on

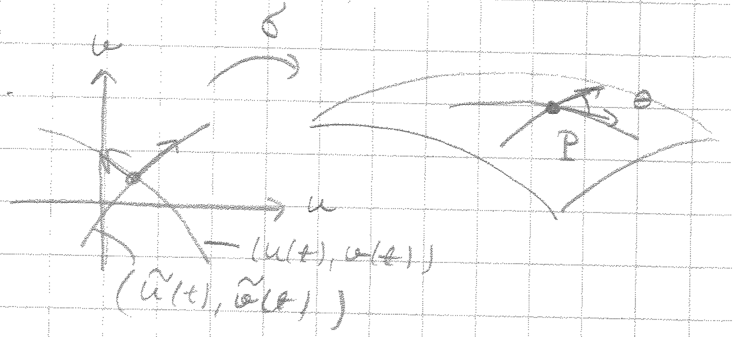
a) taso b) yleistetty sylinteri c) yleistetty kartio

d) käyrän tangentin kehittämä pinta tai niiden osa.

Ol. $t \mapsto \gamma(t) = \sigma(u(t), v(t))$, $t \mapsto \tilde{\gamma}(t) = \sigma(\tilde{u}(t), \tilde{v}(t))$

polkuja, $\sigma : U \to \mathbb{R}^3$ sileän pinnan S sään. parametrisointi

ja $\gamma(t_0) = P = \tilde{\gamma}(t_0) \in \sigma(U)$



Aset. $\theta = \measuredangle(\gamma(t_0), \tilde{\gamma}(t_0)) :=$

$$\measuredangle(\gamma'(t_0), \tilde{\gamma}'(t_0))$$

$$\Rightarrow \cos\theta = \frac{\gamma'(t_0) \cdot \tilde{\gamma}'(t_0)}{\|\gamma'(t_0)\| \, \|\tilde{\gamma}'(t_0)\|}$$

Ketjus $\Rightarrow$ $\gamma' = \sigma_u u' + \sigma_v v'$ $\quad \tilde{\gamma}' = \sigma_u \tilde{u}' + \sigma_v \tilde{v}'$

$$\Rightarrow \gamma' \cdot \tilde{\gamma}' = \sigma_u \cdot \sigma_u \, u'\tilde{u}' + \sigma_u \cdot \sigma_v \, u'\tilde{v}' + \sigma_v \cdot \sigma_u \, v'\tilde{u}' + \sigma_v \cdot \sigma_v \, v'\tilde{v}'$$

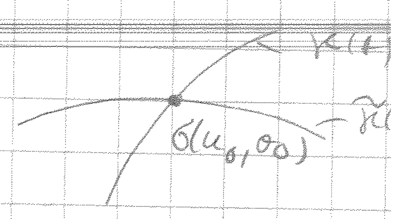
$$= E u'\tilde{u}' + F(u'\tilde{v}' + v'\tilde{u}') + G v'\tilde{v}'$$

$$\Rightarrow \cos\theta = \frac{E u'\tilde{u}' + F(u'\tilde{v}' + v'\tilde{u}') + G v'\tilde{v}'}{(E u'^2 + 2F u'v' + G v'^2)^{1/2} (E \tilde{u}'^2 + 2F \tilde{u}'\tilde{v}' + G \tilde{v}'^2)^{1/2}}$$

Esim koordinaattikäyrille $\gamma(t) = \sigma(u_0, v_0 + t - t_0)$, $\tilde{\gamma}(t) = \sigma(u_0 + t - t_0, v_0)$

saadaan $u(t) = u_0$, $v(t) = v_0 + t - t_0$

$\tilde{u}(t) = u_0 + t - t_0$, $\tilde{v}(t) = v_0$



$\Rightarrow u' \equiv 0 \equiv \tilde{v}'$, $v' \equiv 1 = \tilde{u}'$

$$\Rightarrow \cos\theta = \frac{F}{\sqrt{EG}}$$

Erityisesti $\gamma'(t_0) \perp \tilde{\gamma}'(t_0) \Leftrightarrow F = 0$.

Määritelmä Jos S_1, S_2 sileitä pintoja, niin diffeomorfismi $f : S_1 \to S_2$

on konforminen jos $\measuredangle(\gamma'(t_0), \tilde{\gamma}'(t_0)) = \measuredangle((f\circ\gamma)'(t_0), (f\circ\tilde{\gamma})'(t_0))$

$\forall$ sään $\gamma, \tilde{\gamma}$ s.e. $\gamma(t_0) = \tilde{\gamma}(t_0)$ eli f säilyttää kulmat

Erityisen kiinnostavia ovat sään. parametrisoinnit

$\sigma : U \to \mathbb{R}^3$ jotka konformisia (=konforminen parametrisointi)

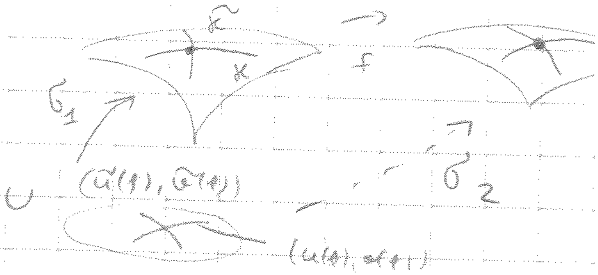
2.3. Lause Diffeomorfismi $f : S_1 \to S_2$ on konforminen

$\Leftrightarrow$ S_1:n parametrisoinnilla σ_1, σ_1:n ja $f \circ \sigma_1$:n

ensimmäiset perusmuodot ovat keskenään (pisteittäin) verrannollisia.

Tod: Lokaali väite $\Rightarrow$ riittää tark. yhtä parametrisointia

$\sigma_1 : U \to \mathbb{R}^3$, $\sigma_2 = f \circ \sigma_1 : U \to \mathbb{R}^3$



"$\Leftarrow$" $E_2\,du^2 + 2F_2\,du\,dv + G_2\,dv^2$

$$\overset{(*)}{=} \lambda(E_1\,du^2 + 2F_1\,du\,dv + G_1\,dv^2),$$

missä (E_1, F_1, G_1) σ_1:n 1. perusmuodon kertoimet
(E_2, F_2, G_2) σ_2 " " "

$\lambda : U \to \mathbb{R}$, $(u,v) \mapsto \lambda(u,v)$ sileä (verrannollisuus)funktio.

$E_i, G_i > 0 \Rightarrow \lambda(u,v) > 0 \quad \forall (u,v)$

Ol. $\gamma, \tilde{\gamma}$ $\sigma_1(U)$:n polkuja $\gamma(t) = \sigma_1(u(t), v(t))$,

$\tilde{\gamma}(t) = \sigma_1(\tilde{u}(t), \tilde{v}(t))$ ja $\gamma(t_0) = \tilde{\gamma}(t_0)$

$t \mapsto (f \circ \gamma)(t) = f(\sigma_1(u(t), v(t))) = \sigma_2(u(t), v(t)) \in \sigma_2(U)$
vast.
$t \mapsto (f \circ \tilde{\gamma})(t) = f(\sigma_2(\tilde{u}(t), \tilde{v}(t))) = \sigma_2(\tilde{u}(t), \tilde{v}(t)) \in \sigma_2(U)$

näiden välinen kulma θ pist. $\sigma_2(u(t_0), v(t_0)) = \sigma_2(u(t_0), v(t_0))$:

$$\cos\theta = \frac{E_2 u'\tilde{u}' + F_2(u'\tilde{v}' + v'\tilde{u}') + G_2 v'\tilde{v}'}{(E_2 u'^2 + 2F_2 u'v' + G_2 v'^2)^{1/2}(E_2\tilde{u}'^2 + 2F_2\tilde{u}'\tilde{v}' + G_2\tilde{v}'^2)^{1/2}}$$

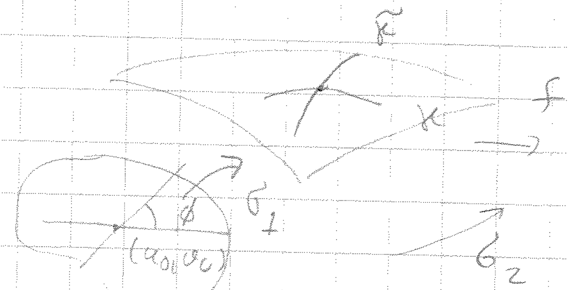
$$\overset{(*)}{=} \frac{E_1 u'\tilde{u}' + F_1(u'\tilde{v}' + v'\tilde{u}') + G_1 v'\tilde{v}'}{(E_1 u'^2 + 2F_1 u'v' + G_1 v'^2)^{1/2}(E_1\tilde{u}'^2 + 2F_1\tilde{u}'\tilde{v}' + G_1\tilde{v}'^2)^{1/2}} = \measuredangle(\tilde{\gamma}'(t_0), \gamma'(t_0))$$

$\Rightarrow$ f konf.

"$\Rightarrow$" Pitää os: Jos $\measuredangle(\tilde\gamma'(t_0), \gamma'(t_0)) = \measuredangle((f\circ\tilde\gamma)'(t_0), (f\circ\gamma)'(t_0))$ II 2

$\forall\; t\mapsto\gamma(t) = \sigma_1(u(t), v(t))$, $t\mapsto\tilde\gamma(t) = \sigma_1(\tilde u(t), \tilde v(t))$ s.e. $\gamma(t_0) = \tilde\gamma(t_0)$

niin perusmuodot verrannollisia.



Olk. $(u_0, v_0) \in U$ ja

$$\gamma(t) = \sigma_1(u_0 + t, v_0)$$

$$\tilde\gamma(t) = \sigma_1(u_0 + t\cos\phi, v_0 + t\sin\phi) \qquad \gamma(0) = \sigma_1(u_0, v_0) = \tilde\gamma(0)$$

I. $u(t) = u_0 + t$, $v(t) \equiv v_0$, $\tilde u(t) = u_0 + t\cos\phi$, $\tilde v(t) = v_0 + t\sin\phi$

$\Rightarrow$ $u'(t) \equiv 1$, $v'(t) \equiv 0$, $\tilde u'(t) = \cos\phi$, $\tilde v'(t) = \sin\phi$

$$\Rightarrow \cos\measuredangle(\gamma'(0), \tilde\gamma'(0)) = \frac{E_1\cos\phi + F_1\sin\phi}{\left[E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)\right]^{1/2}}$$

el. ‖

$$\cos\measuredangle((f\circ\gamma)'(0), (f\circ\tilde\gamma)'(0)) = \frac{E_2\cos\phi + F_2\sin\phi}{\left[E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)\right]^{1/2}}$$

$$\Leftrightarrow (E_1\cos\phi + F_1\sin\phi)^2 E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)$$

$$= (E_2\cos\phi + F_2\sin\phi)^2 E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)$$

Korj. $(E_1\cos\phi + F_1\sin\phi)^2 = E_1^2\cos^2\phi + 2E_1F_1\cos\phi\sin\phi + F_1^2\sin^2\phi$

$$= E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi) + (F_1^2 - E_1G_1)\sin^2\phi$$

ja $(E_2\cos\phi + F_2\sin\phi)^2 = E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi) + (F_2^2 - E_2G_2)\sin^2\phi$

$$\Rightarrow \left(\underbrace{E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)} + (F_1^2 - E_1G_1)\sin^2\phi\right)E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)$$

$$= \left(\underbrace{E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi)} + (F_2^2 - E_2G_2)\sin^2\phi\right)E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)$$

$$\Leftrightarrow (F_1^2 - E_1G_1)E_2(E_2\cos^2\phi + 2F_2\cos\phi\sin\phi + G_2\sin^2\phi) =$$

$$(F_2^2 - E_2G_2)E_1(E_1\cos^2\phi + 2F_1\cos\phi\sin\phi + G_1\sin^2\phi)$$

I 2.12

$\Leftrightarrow \Big((F_1^2 - E_1G_1)E_2^2 - (F_2^2 - E_2G_2)E_1^2 \Big)\cos^2\phi +$

$2\Big[(F_1^2 - E_1G_1)E_2F_2 - (F_2^2 - E_2G_2)E_1F_1 \Big)\cos\phi\sin\phi +$ (*)

$\Big((F_1^2 - E_1G_1)E_2G_2 - (F_2^2 - E_2G_2)E_1G_1 \Big)\sin^2\phi = 0$

$\phi = 0 \Rightarrow (F_1^2 - E_1G_1)E_2^2 - (F_2^2 - E_2G_2)E_1^2 = 0$ a)

$\phi = \frac{\pi}{2} \Rightarrow F_1^2E_2G_2 - F_2^2E_1G_1 = 0$ b)

$(*) \Rightarrow (F_1^2 - E_1G_1)E_2F_2 - (F_2^2 - E_2G_2)E_1F_1 = 0$ c)

$F_i^2 - E_iG_i \neq 0$, sillä jos $F_i^2 = E_iG_i$ niin

$\|\sigma_u \cdot \sigma_v\|^2 = \|\sigma_u\|^2\|\sigma_v\|^2 \Rightarrow \sigma_u \parallel \sigma_v$ mutta σ_i sään $\lightning$.

Merk. $\lambda = \dfrac{(F_2^2 - E_2G_2)E_1}{(F_1^2 - E_1G_1)E_2} \overset{a)}{\Rightarrow} E_2 = \lambda E_1$

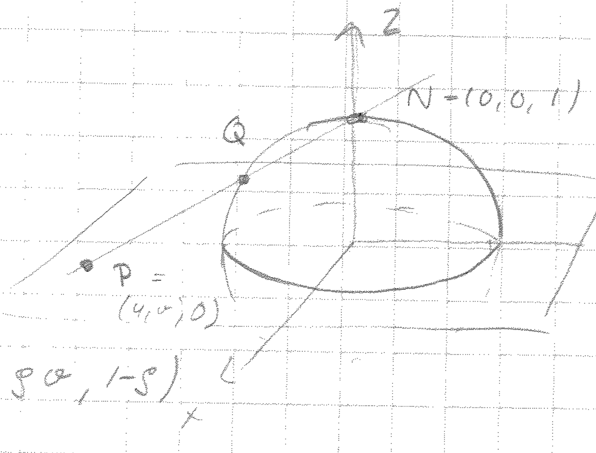
$c) \Rightarrow F_2 = \lambda F_1$, $\quad b) \Rightarrow F_1^2\lambda E_1G_2 = \lambda^2F_1^2E_1G_1$

$\overset{\lambda, E_1 > 0}{\Rightarrow} F_1^2G_2 = \lambda F_1^2G_1$ jos $F_1 \neq 0 \Rightarrow G_2 = \lambda G_1$

jos $F_1 = 0 \Rightarrow F_2 = 0 \overset{a)}{\Rightarrow} E_1G_1(\lambda E_1)^2 = \lambda E_1G_2E_1^2$

$\overset{E_1, \lambda > 0}{\Rightarrow} G_2 = \lambda G_1$. $\square$

Esim. stereografinen projektio



Ol. $S = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$

$\exists\, \rho \in \mathbb{R}$ s.t. $\bar{Q} - \bar{N} = \rho(\bar{P} - \bar{N})$

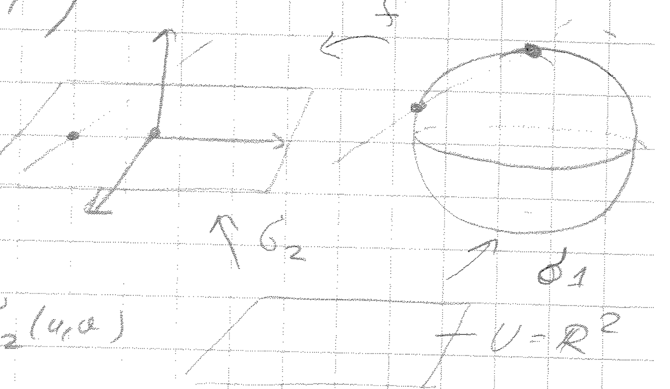
$\Rightarrow Q = (0,0,1) + \rho\big((u,v,0) - (0,0,1)\big) = (\rho u, \rho v, 1-\rho)$

$Q \in S \Rightarrow \rho^2u^2 + \rho^2v^2 + (1-\rho)^2 = 1 \Leftrightarrow \rho^2(u^2+v^2+1) - 2\rho = 0$

$\Leftrightarrow (\rho = 0 \;\vee)\; \rho = \dfrac{2}{u^2+v^2+1}$

(II 2..

$$\Rightarrow \bar{\varphi} = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right) =: \sigma_1(u,v)$$



$\sigma_1 : \mathbb{R}^2 \to S \setminus \{N\}$

Aset.

$\sigma_2 : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma_2(u,v) = (u,v,0)$

$f : S \setminus \{N\} \to \mathbb{R}^2$, $\sigma_1(u,v) \longmapsto \sigma_2(u,v)$

on stereografinen projektio

V: f konforminen.

Riittää os L.2.3. $\Rightarrow$ σ_2:n ens. perusmuoto verrannollinen

$\sigma_2 = f \circ \sigma_1$:n 1. perusmuotoon $du^2 + dv^2$.

$$\sigma_{1u} = \frac{1}{(u^2+v^2+1)^2} \left(2(u^2+v^2+1) - 2u \cdot 2u, \; -4uv, \; 2u(u^2+v^2+1) - 2u(u^2+v^2-1) \right)$$

$$= \frac{1}{(u^2+v^2+1)^2} \left(2(v^2-u^2+1), -4uv, 4u \right)$$

$$\sigma_{1v} = \frac{1}{(u^2+v^2+1)^2} \left(-4uv, \; 2(u^2+v^2+1) - 4v^2, \; 2v(u^2+v^2+1) - 2v(u^2+v^2-1) \right)$$

$$= \frac{1}{(u^2+v^2+1)^2} \left(-4uv, 2(u^2-v^2+1), 4v \right)$$

$$\Rightarrow F_1 = \sigma_{1u} \cdot \sigma_{1v} = \frac{1}{(u^2+v^2+1)^4} \left(-8uv(v^2-u^2+1) - 8uv(u^2-v^2+1) + 16uv \right) = 0$$

$$E_1 = \sigma_{1u} \cdot \sigma_{1u} = \frac{1}{(u^2+v^2+1)^4} \left(4(v^2-u^2+1)^2 + 16u^2v^2 + 16u^2 \right)$$

$$= \frac{1}{(u^2+v^2+1)^4} \left(4\left(v^4 + u^4 - 2v^2u^2 + 1 + 2(v^2-u^2) \right) + 16u^2v^2 + 16u^2 \right)$$

$$= \frac{1}{(u^2+v^2+1)^4} \left(\underbrace{4(v^4+u^4+1) + 8u^2v^2 + 8u^2 + 8v^2}_{= 4(u^2+v^2+1)^2} \right) = \frac{4}{(u^2+v^2+1)^2}$$

symm. $u,v \Rightarrow$ (II 2.1

$$G_1 = \sigma_{1v}\cdot\sigma_{1v} = \frac{4}{(u^2+v^2+1)^2}$$

$$\therefore \quad E_1\,du^2 + 2F_1\,du\,dv + G_1\,dv^2 = \frac{4}{(u^2+v^2+1)^2}(du^2+dv^2) =: \lambda(u,v)(du^2+dv^2)$$

□

<u>Määritelmä</u> Alueen $R \subset U$ <u>pinta-ala</u> pintaelementillä $\sigma: U \to \mathbb{R}^3$ on $A_\sigma(R) = \iint_R \|\sigma_u \times \sigma_v\|\,du\,dv$.

<u>2.4. Lause</u> $\|\sigma_u \times \sigma_v\| = (EG-F^2)^{1/2}$

<u>Tod:</u> $\|\sigma_u \times \sigma_v\|^2 = \|\sigma_u\|^2\|\sigma_v\|^2 \sin^2 \measuredangle(\sigma_u,\sigma_v)$

$= \|\sigma_u\|^2\|\sigma_v\|^2(1-\cos^2\measuredangle(\sigma_u,\sigma_v))$

$= \|\sigma_u\|^2\|\sigma_v\|^2 - (\sigma_u\cdot\sigma_v)^2 = EG-F^2$. □.

<u>Huom</u> $EG-F^2>0$, sillä σ sään.

Kirj. $dA_\sigma = (EG-F^2)^{1/2}\,du\,dv$ infinitesimaalinen pinta-alan muutos.

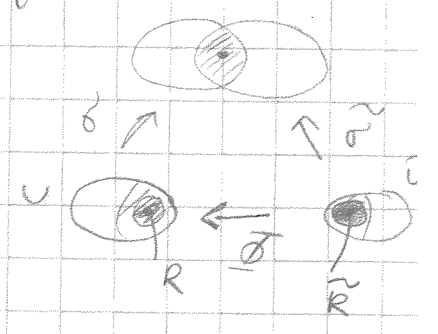
~~2.5. Lause Pinta-ala on riippumaton parametrisoinnista.~~

<u>Tod</u>: Ol. $\sigma: U \to \mathbb{R}^3$, $\tilde\sigma: \tilde U \to \mathbb{R}^3$ parametrisointeja s.e.

$\sigma(U)\cap\tilde\sigma(\tilde U) \neq \emptyset$. Olk. $\Phi = \sigma^{-1}\circ\tilde\sigma$ ja

$\tilde R \subset \tilde\sigma^{-1}(\sigma(U)\cap\tilde\sigma(\tilde U))$. Aik. L. II.1.1. tod $\Rightarrow$

$\tilde\sigma_{\tilde u} \times \tilde\sigma_{\tilde v} = J_\Phi\, \sigma_u \times \sigma_v$. Merk. $R = \Phi(\tilde R)$



$$\Rightarrow \iint_{\tilde R} \|\tilde\sigma_{\tilde u} \times \tilde\sigma_{\tilde v}\|\,d\tilde u\,d\tilde v = \iint_{\tilde R} |J_\Phi|\,\|\sigma_u \times \sigma_v\|\,d\tilde u\,d\tilde v$$

$$= \iint_R \|\sigma_u \times \sigma_v\|\,du\,dv$$

↑
muut. vaihto $(u,v) = \Phi(\tilde u,\tilde v)$

□.

<u>Määritelmä</u> Ol. S_1, S_2 sileitä pintoja. Diffeomorfismi $f: S_1 \to S_2$ on <u>pinta-alan säilyttävä</u>, jos pätee

$\mathcal{A}_\sigma(R) = \mathcal{A}_{f\circ\sigma}(R) \quad \forall\ R \subset U$, $\sigma: U \to \mathbb{R}^3$ S_1:n parametrisointi.

<u>2.6. Lause</u> Diffeomorfismi $f: S_1 \to S_2$ on pinta-alan säilyttävä jos ja vain jos pätee $E_1G_1 - F_1^2 = E_2G_2 - F_2^2$ kun $E_1du^2 + 2F_1du\,dv + G_1dv^2$ on S_1:n parametrisoinnin $\sigma_1: U \to \mathbb{R}^3$ 1. perusmuoto ja $E_2du^2 + 2F_2du\,dv + G_2dv^2$ on S_2:n parametrisoinnin $\sigma_2 = f\circ\sigma_1: U \to \mathbb{R}^3$ 1. perusmuoto.

<u>Tod</u>: "$\Leftarrow$" Jos $R \subset U$, niin L.2.4: $\mathcal{A}_{\sigma_1}(R) = \iint_R (E_1G_1 - F_1^2)^{1/2}\,du\,dv$

ja $\mathcal{A}_{\sigma_2}(R) = \iint_R (E_2G_2 - F_2^2)^{1/2}\,du\,dv$ ja

$f\circ\sigma_1(R) = \sigma_2(R) \Rightarrow \mathcal{A}_{\sigma_1}(R) = \mathcal{A}_{f\circ\sigma_1}(R)$.

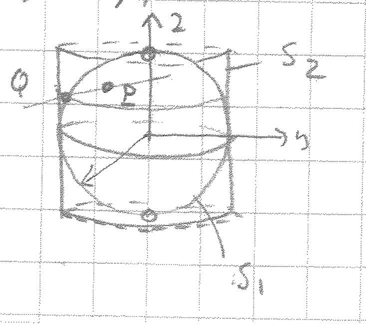
"$\Rightarrow$" Ol. $f: S_1 \to S_2$ pinta-alan säilyttävä $\Rightarrow \forall\ R \subset U$

$\mathcal{A}_{\sigma_1}(R) = \mathcal{A}_{f\circ\sigma_1}(R) \Rightarrow \iint_R (E_1G_1 - F_1^2)^{1/2}\,du\,dv = \iint_R (E_2G_2 - F_2^2)^{1/2}\,du\,dv$

$\Rightarrow E_1G_1 - F_1^2 = E_2G_2 - F_2^2$ $\square$.

1° <u>Esim</u> Ol. $f: \underbrace{S^2(0,1) \setminus \{(0,0,\pm 1)\}}_{=S_1} \to \{(x,y,z) \mid x^2+y^2=1,\ z \in (-1,1)\} =: S_2$

kuvaus, joka vie pisteen $P = (x,y,z) \in S_1$ pisteeksi $Q = (\bar{x},\bar{y},\bar{z}) \in S_2$, joka lähimpänä pistettä P s.e. $\mathrm{sp}\{PQ\} \cap \{(0,0,z)\} \neq \emptyset$



ja $\mathrm{sp}\{PQ\} \parallel xy$-taso

$\Rightarrow Z = z \,,\quad (X, Y) = \lambda(x,y)$ jollakin $\lambda \in \mathbb{R}$

$Q \in S_2 \Rightarrow 1 = X^2 + Y^2 = \lambda^2(x^2+y^2) \Rightarrow \lambda = (\pm 1)(x^2+y^2)^{-\frac{1}{2}}$

$$\Rightarrow f(x,y,z) = \left(\frac{x}{(x^2+y^2)^{1/2}}, \frac{y}{(x^2+y^2)^{1/2}}, z \right).$$

V: f on pinta-alan säilyttävä diffeomorfismi

Val. S_1:n kartasto $(\sigma_1, U_1), (\sigma_1, U_2)$

$\sigma_1(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \qquad U_1 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$

$\qquad U_2 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi)$

$\sigma_2(\theta,\varphi) := (f \circ \sigma_1)(\theta, \varphi) = (\cos\varphi, \sin\varphi, \sin\theta)$

$\Rightarrow (\sigma_2, U_1), (\sigma_2, U_2)$ S_2:n kartasto

aik. Esim 2° s. 2.4 $\Rightarrow E_1 = 1, F_1 = 0, G_1 = \cos^2\theta$

σ_2:n 1. perusmuoto: $\sigma_{2\theta} = (0, 0, \cos\theta), \; \sigma_{2\varphi} = (-\sin\varphi, \cos\varphi, 0)$

$E_2 = \|\sigma_{2\theta}\|^2 = \cos^2\theta \qquad F_2 = 0, \quad G_2 = 1$

$\Rightarrow E_1G_1 - F_1^2 = E_2G_2 - F_2^2 \overset{L.2.6}{\Rightarrow}$ kuvaus

$f: S_1 \to S_2 \quad (\theta, \varphi) \mapsto (\theta, \varphi)$ diffeo (!) ja

pinta-alan säilyttävä.

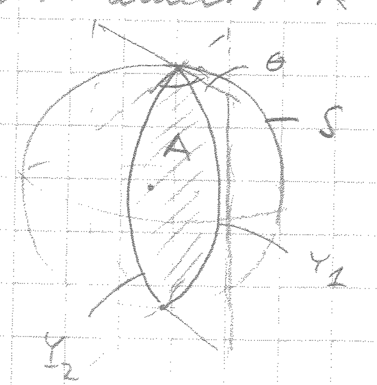
2° Esim Pallon $S^2(0,1)$ kahden isoympyrän väliin jäävän alueen A

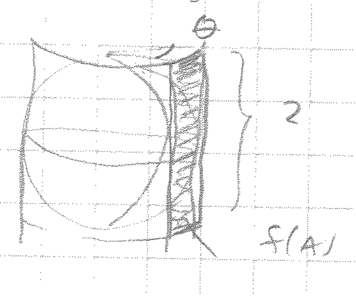
pinta-ala? Voi ol. $\Sigma_1 \cap Y_2 = \{(0,0,\pm 1)\}$

(palloa pyörittämällä). Olk. $\measuredangle(\Sigma_1, \Sigma_2) = \theta$

esim 1° $\Rightarrow f(A) =$

Area$(A) = 2\theta$



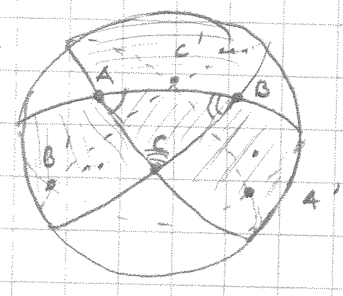


2.7. Lause Ol. ABC kolmio pallolla $S^2(0,1)$, jonka sivut ovat isoympyrän kaaria. Kolmion pinta-ala on

$$\measuredangle A + \measuredangle B + \measuredangle C - \pi$$

missä $\measuredangle A$, $\measuredangle B$, $\measuredangle C$ pisteisiin A B C liittyvät kulmat.

Tod: Olk kolmen isoympyrän leikkauspisteet A, B, C ja A', B', C' (antipodaaliset pisteet) jakaa pallon pinnan 8:n kolmioon



Esim 2° $\Rightarrow$ $\mathcal{A}(ABC) + \mathcal{A}(BCA') = 2\measuredangle A$

$$\mathcal{A}(ABC) + \mathcal{A}(ACB') = 2\measuredangle B$$

$$\mathcal{A}(ABC) + \mathcal{A}(ABC') = 2\measuredangle C$$

Puolipallo muodostuu kolmioista ABC ABC' $A'BC$ $A'BC'$

$\Rightarrow$ $\mathcal{A}(ABC) + \mathcal{A}(ABC') + \mathcal{A}(A'BC) + \mathcal{A}(A'BC') = 2\pi$

---

antipodaalinen kuvaus $a: x \mapsto -x$, $x \in S^2(0,1)$ on isometria (ja siis säilyttää pinta-alan (HT)) $\Rightarrow$

$\mathcal{A}(A'BC') = \mathcal{A}(AB'C)$

$$\Rightarrow 2\mathcal{A}(ABC) + \underbrace{\{\mathcal{A}(ABC) + \mathcal{A}(BCA') + \mathcal{A}(ACB') + \mathcal{A}(ABC')\}}_{=2\pi}$$

$$= 2(\measuredangle A + \measuredangle B + \measuredangle C)$$

$$\Rightarrow \mathcal{A}(ABC) = \measuredangle A + \measuredangle B + \measuredangle C - \pi. \quad \square$$