Helsinki University of Technology, TKK
Mat-5.3701 Analytical Mechanics
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Exercise 6, 2006 February 28, Tuesday 14-16 Y313
`http://math.tkk.fi/teaching/analmek`

1. The motion of a particle (mass m) is described by the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2m} + m\alpha q_1 \,,$$

where α is a constant. Construct the solution of the equations of motion for the initial conditions

$$q_1(0) = x_0 \,, \quad q_2(0) = y_0 \,, \quad p_1(0) = p_x \,, \quad p_2(0) = p_y \,,$$

making use of the Poisson brackets.

2. For a three body system with masses m_i, coordinates $\boldsymbol{r}_i$, and momenta $\boldsymbol{p}_i$ we introduce the following Jacobian coordinates:

$$\begin{aligned}
\boldsymbol{\varphi}_1 &:= \boldsymbol{r}_2 - \boldsymbol{r}_1 \,, \\
\boldsymbol{\varphi}_2 &:= \boldsymbol{r}_3 - \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \,, \\
\boldsymbol{\varphi}_3 &:= \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3}{m_1 + m_2 + m_3} \,, \\
\boldsymbol{\pi}_1 &:= \frac{m_1\boldsymbol{p}_2 - m_2\boldsymbol{p}_1}{m_1 + m_2} \,, \\
\boldsymbol{\pi}_2 &:= \frac{(m_1 + m_2)\boldsymbol{p}_3 - m_3(\boldsymbol{p}_1 + \boldsymbol{p}_2)}{m_1 + m_2 + m_3} \,, \\
\boldsymbol{\pi}_3 &:= \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 \,.
\end{aligned}$$

   (i) What is the physical interpretation of the new momenta $\boldsymbol{\pi}_i$?

   (ii) How would you define the corresponding coordinates for four or more particles?

   (iii) Show in at least two (equivalent) ways that the transformation

$$(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \to (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3)$$

   is canonical.

3. **Home exercise:** Consider the problem of a simple pendulum (one arm and one mass). Write down

   (i) the Lagrange equations of motion,

   (ii) Hamilton's equations of motion and

   (iii) the Hamilton–Jacobi partial differential equations.

   Try what Matlab, Maple or Mathematica gives for solutions to these.