Helsinki University of Technology, TKK
Mat-5.3701 Analytical Mechanics
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Exercise 4, 2006 February 14, Tuesday 14-16 Y313
`http://math.tkk.fi/teaching/analmek`

1. Work out the calculations for the case D in the section 2.23 of canonical transformations.

2. (i) Show that canonical transformations leave the physical dimension of the product $p_i q_i$ unchanged, i.e. $[P_i Q_i] = [p_i q_i]$. Let Φ be the generating function for a canonical transformation. Show that

$$[p_i q_i] = [P_k Q_k] = [\Phi] = [Ht]\,,$$

where H is the Hamiltonian and t the time.

(ii) In the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$ of the harmonic oscillator introduce the variables

$$x_1 := \omega\sqrt{m}q\,, \quad x_2 := p/\sqrt{m}\,, \quad \tau := \omega t\,,$$

thus obtaining $H = (x_1^2 + x_2^2)/2$. What is the generating function $\widehat{\Phi}(x_1, y_1)$ for the canonical transformation $x \to y$ that corresponds to the function $\Phi(q, Q) = (m\omega q^2/2)\cot Q$? Calculate the matrix $M_{ik} = \partial x_i / \partial y_k$ and confirm that $\det \boldsymbol{M} = 1$ and $\boldsymbol{M}^T \boldsymbol{J} \boldsymbol{M} = \boldsymbol{J}$.

3. **Home exercise:** The group Sp_{2f} is particularly simple for $f = 1$, i.e. in two dimensions.

(i) Show that every matrix

$$\boldsymbol{M} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is symplectic if and only if $a_{11}a_{22} - a_{12}a_{21} = 1$.

(ii) Therefore, the orthogonal matrices

$$\boldsymbol{O} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

and the symmetric matrices

$$\boldsymbol{S} = \begin{pmatrix} x & y \\ y & z \end{pmatrix}\,,$$

with $xz - y^2 = 1$, belong to Sp_{2f}. Show that every $\boldsymbol{M} \in Sp_{2f}$ can be written as a product

$$\boldsymbol{M} = \boldsymbol{S}\boldsymbol{O}\,,$$

of a symmetric matrix $\boldsymbol{S}$ and an orthogonal matrix $\boldsymbol{O}$.