Helsinki University of Technology, TKK
Mat-5.3701 Analytical Mechanics
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Exercise 3, 2006 February 7, Tuesday 14-16 Y313
`http://math.tkk.fi/teaching/analmek`

1. Try to find another proof (than in the text book) for the fact

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \exp(-\varphi \boldsymbol{M}),$$

with

$$\boldsymbol{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2. (i) Let the Lagrangian be

$$L(q_1, q_2, \dot q_1, \dot q_2, t) = T - U,$$
$$T = \sum_{i,k=1}^{2} c_{ik}\dot q_i \dot q_k + \sum_{k=1}^{2} b_k \dot q_k + a.$$

Under what conditions can one construct the Hamiltonian $H(q,p,t)$ and what are the generalized momenta p_1, p_2 and H? Confirm that the Legendre transform of H is again L and that

$$\det\left(\frac{\partial^2 L}{\partial \dot q_k \partial \dot q_i}\right)\det\left(\frac{\partial^2 H}{\partial p_n \partial p_m}\right) = 1.$$

Hint: Take $d_{11} = 2c_{11}$, $d_{12} = d_{21} = c_{12} + c_{21}$, $d_{22} = 2c_{22}$, $\pi_i = p_i - b_i$.

(ii) Assume now that $L = L(x_1 \equiv \dot q_1, x_2 \equiv \dot q_2, q_1, q_2, t) \equiv L(x_1, x_2, \boldsymbol{u})$, with $\boldsymbol{u} \equiv (q_1, q_2, t)$, is an arbitrary Lagrangian function. We expect the momenta $p_i = p_i(x_1, x_2, \boldsymbol{u})$ derived from L to be independent functions of x_1 and x_2, i.e. that there is no function $F(p_1(x_1, x_2, \boldsymbol{u}), p_2(x_1, x_2, \boldsymbol{u}))$ that vanishes identically.
Show that if p_1 and p_2 were dependent the determinant of the second order derivatives of L with respect to x_i would vanish.
Hint: Consider $\frac{dF}{dx_1}$ and $\frac{dF}{dx_2}$.

3. **Home exercise:** A particle of mass m can move in one dimension under the influence of two springs connected to fixed points a distance a apart (see Figure 1). The springs obey Hooke's law and have zero unstreched lengths and force constants k_1 and k_2, respectively.

(i) Using the position of the particle from one fixed point as the generalized coordinate q find the Lagrangian and the corresponding Hamiltonian. Is the energy conserved? Is the Hamiltonian conserved?

(ii) Introduce a new coordinate Q defined by

$$Q = q - b\sin\omega t, \quad b = \frac{k_2 a}{k_1 + k_2}.$$

What is the Lagrangian in terms of Q? What is the corresponding Hamiltonian? Is the energy conserved? Is the Hamiltonian conserved?

4. Prove that Hamilton's equations can alternatively be obtained as the extremals of

$$\int_{t_1}^{t_2} \Big(-\sum_{k=1}^{f} \dot p_k q_k - H(q_1, ..., q_f, p_1, ..., p_f, t)\Big)dt.$$

(Note that the integrand is obtained from the Lagrangian by subtracting $\frac{d}{dt}\sum_{k=1}^{f} p_k q_k$.)

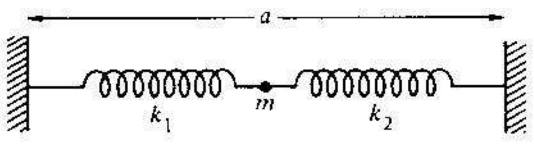

Figure 1: A moving mass with springs