Helsinki University of Technology, TKK
Mat-5.3701 Analytical Mechanics
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Exercise 2, 2006 January 31, Tuesday 14-16 Y313
`http://math.tkk.fi/teaching/analmek`

1. Reverse the Legendre transformation to derive the properties of $L(q_i, \dot{q}_i, t)$ from $H(q_i, p_i, t)$ treating $\dot{q}_i$ as independent quantities, and show that it leads to the Lagrangian equations of motion.

2. **Home exercise:** The Lagrangian for a system can be written as
$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2}\,,$$
where a, b, c, f, g and k are constants. What is the Hamiltonian? What quantities are conserved?

3. With respect to an inertial system $\boldsymbol{K}_0$ the Lagrangian function of a particle is
$$L_0 = \frac{1}{2}m\dot{\boldsymbol{x}}_0{}^2 - U(\boldsymbol{x}_0)\,.$$
The frame of reference $\boldsymbol{K}$ has the same origin as $\boldsymbol{K}_0$ but rotates about the latter with a constant angular velocity $\boldsymbol{\omega}$. Show that the Lagrangian with respect to $\boldsymbol{K}$ reads
$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 + m\dot{\boldsymbol{x}}\cdot(\boldsymbol{\omega}\times\boldsymbol{x}) + \frac{1}{2}m(\boldsymbol{\omega}\times\boldsymbol{x})^2 - U(\boldsymbol{x})\,.$$
Derive the equations of motion of the section 1.25 from this.

4. *Invariance under time translations and Noether's theorem*:
The theorem of E. Noether can be applied to the case of translations of *time* by means of the following trick: Make t a coordinate-like variable by parametrizing both $\boldsymbol{q}$ and t by $\boldsymbol{q} = \boldsymbol{q}(\tau)$, $t = t(\tau)$ and by defining the following Lagrangian function:
$$\bar{L}(\boldsymbol{q}, t, \frac{d\boldsymbol{q}}{d\tau}, \frac{dt}{d\tau}) := L(\boldsymbol{q}, \frac{1}{dt/d\tau}\frac{d\boldsymbol{q}}{d\tau}, t)\frac{dt}{d\tau}\,.$$
   (i) Show that Hamilton's variational principle applied to $\bar{L}$ yields the same equations of motion as for L.
   (ii) Assume L to be invariant under time translations
$$h_s(\boldsymbol{q}, t) = (\boldsymbol{q}, t+s)\,.$$
   Apply Noether's theorem to $\bar{L}$ and find the constant of the motion corresponding to the invariance above.