TKK Matematiikan laitos Arponen/Pursiainen

Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 8 (27.3.2008)

These are held in the computer classroom Y339b (close to Y313). Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise. In addition to that, hand in the exercises marked with [Comp. hand-in] in the *next* exercise session (3rd April, that is).

* 1. Show that if A is "strictly column diagonally dominant", that is,

$$|a_{kk}| > \sum_{j \neq k} |a_{jk}| \quad \forall k,$$

then Gaussian elimination with partial pivoting does not produce any row interchanges, i.e. all the permutation matrices $P_i = I$.

2. Confirm the results of Exercise 7, question 3 on the sparsity patterns of the L, U factors of a banded matrix A. You will need the `spy` command. (`help subplot` might be nice as well.) For (a) part, make your A diagonally dominant to make sure the LU without pivoting exists.

* 3. Suppose $A \in \mathbb{R}^{m \times m}$ is a unit lower triangular apart from the last column which is all ones, and below the diagonal each element is -1, i.e. A has the form

$$A = \begin{pmatrix} 1 & & & & & 1 \\ -1 & 1 & & & & 1 \\ -1 & -1 & 1 & & & 1 \\ \vdots & & \ddots & \ddots & & 1 \\ -1 & -1 & \dots & -1 & 1 & 1 \\ -1 & -1 & \dots & & -1 & 1 \end{pmatrix}.$$

- Show that the growth factor $\rho(A) = 2^{m-1}$.
- Describe how the Gaussian elimination (with partial pivoting) simplifies for this type of A.

4. [Comp. hand-in] Let $A = B - 2I$ where $B \in \mathbb{R}^{10 \times 10}$ with random entries from the standard normal distribution (`help randn`). Write a program to plot $\|e^{tA}\|_2$ against t for $0 \leq t \leq 20$ on a log scale, comparing the result to the straight line $e^{t\alpha(A)}$ where $\alpha(A) = \max_j \text{Re}\,(\lambda_j)$ is the *spectral abscissa* of A and λ_j's are the eigenvalues of A. Run the program for 10 matrices A and comment on the results. What property of a matrix leads to a $\|e^{tA}\|_2$ curve that remains oscillatory as $t \to \infty$?

* 5. Discretizations of differential equations. We will study the 1-dimensional Poisson equation:

$$-v''(x) = f(x), \quad 0 < x < 1, \quad v(0) = v(1) = 0$$

where f is a given function and the primes denote derivatives w.r.t. x. We *discretize* this by replacing $0 < x < 1$ by finitely many evenly spaced points $x_j := jh$ where $h = \frac{1}{N+1}$ and $0 \le j \le N+1$ (i.e. N interior points $+2$ boundary points). Denote $v_i := v(x_i)$ and $f_i := f(x_i)$. Approximate

$$\begin{aligned} v'((j - \frac{1}{2})h) &\approx \frac{v_i - v_{i-1}}{h} \\ v'((j + \frac{1}{2})h) &\approx \frac{v_{i+1} - v_i}{h} \\ v''(x_j) &\approx \frac{v'((j+\frac{1}{2})h) - v'((j-\frac{1}{2})h)}{h} = \dots \end{aligned}$$

(a) Write the discretized version of the Poisson equation as a matrix equation $T_N\, v = b$ where $v := \begin{pmatrix} v_1 & v_2 & \dots & v_N \end{pmatrix}^T$. Why can we drop v_0 and v_{N+1}? What are T_N and b?

(b) Prove: the eigenvalues of T_N are $\lambda_j = 2(1 - \cos \frac{\pi j}{N+1})$. The eigenvectors are z_j, where $(z_j)_k = \sin \frac{jk\pi}{N+1}$.

(c) Approximate, for large N, the sizes of the largest and smallest eigenvalues.

(d) Compute an SVD of T_N.

6. [Comp. hand-in] (Hand in (a),(c),(d).) We continue from question 5.

(a) Write a code for solving $T_N\, v = b$ for a given N and $f = (f_1 \ \dots \ f_N)^T$.

(b) Take $f(x) \equiv 0.7$, a constant function. Use your code to solve the system for different N, e.g. $N = 100, 500, 1000$ (depends on your system what are suitable values). Plot your solution v. Is the shape what you would expect?

(c) Take

$$f(x) = \begin{cases} -a, & 0 < x \le c/10 \\ b, & 1 > x > c/10 \end{cases}$$

where a, b, c are the last three digits of your student ID (choose $c \neq 0$). Solve again for v and plot the result.

(d) Modify your code: use the timing commands `tic, toc` to measure elapsed time. Then, create a sparse version of T_N (`help sparse`) and solve again. Do you notice any speed-up? How about savings in memory? (`whos`)