TKK Matematiikan laitos Arponen/Pursiainen

Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 5 (21.2.2008)

Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise.

1. Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m \times 2m$ hermitian block matrix

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

* 2. Let $A \in \mathbb{C}^{m\times n}$. A more general definition for the pseudoinverse of A (also known as the Moore-Penrose pseudoinverse) is

$$A^+ = \lim_{\epsilon\to 0^+} (A^*A + \epsilon I)^{-1} A^*$$

where I is the identity matrix of suitable size. Show that the limit always exists. Hint: it suffices to show that $\lim_{\epsilon\to 0^+} (A^*A + \epsilon I)^{-1} A^* v$ exists $\forall v \in \mathbb{C}^m$. Consider first the cases $v \in \mathcal{N}(A^*)$ and $v \in \mathcal{R}(A)$.

3. Determine the (a) eigenvalues, (b) determinant, and (c) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as an algebraic proof.

* 4. (Givens' rotations.) Consider the 2×2 orthogonal matrices

$$F = \begin{pmatrix} -c & s \\ s & c \end{pmatrix}, \qquad J = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

where $s = \sin\theta$ and $c = \cos\theta$ for some θ. Here F is a special case of a Householder reflector, while J is a rotation ($\det J = 1$), also known as a *Givens rotation.*

(a) In $\mathbb{R}^2$, describe geometrically what are the relations between v and Fv, and between v and Jv where $v \in \mathbb{R}^2$ is arbitrary.

(b) Describe an algorithm for QR factorization that is analogous to the Householder algorithm, but using Givens rotations instead of the Householder reflection. Which (Householder or Givens) has greater operation count?

5. Suppose the $m \times n$ matrix A has the form

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

where A_1 is a square $n \times n$ invertible matrix and A_2 is $(m-n)\times n$. Prove that $\|A^+\|_2 \le \|A_1^{-1}\|_2$.

* 6. Show that, as $\epsilon \to 0$,

$$\begin{aligned} (1+O(\epsilon))(1+O(\epsilon)) &= 1+O(\epsilon) \\ (1+O(\epsilon))^{-1} &= 1+O(\epsilon). \end{aligned}$$