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The Nonequispaced FFT and its Applications

A Mini Course at Helsinki University of Technology

http://math.tkk.fi/numericsyear/NFFT

Lab 1 **Basics and Matlab**

Exercise 1:
Let $f : \mathbb{R} \to \mathbb{C}$ and its Fourier transform

$$\hat{f}(k) := \int_{\mathbb{R}} f(x) \mathrm{e}^{-2\pi \mathrm{i} k x} \mathrm{d}x, \qquad k \in \mathbb{R},$$

obey $|f(x)| \leq (1+|x|)^{-1-\varepsilon}$ and $|\hat{f}(k)| \leq (1+|k|)^{-1-\delta}$ for some $\delta, \varepsilon > 0$, respectively. Prove, that the Poisson summation formula

$$\sum_{r \in \mathbb{Z}} f(x+r) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \, \mathrm{e}^{2\pi \mathrm{i} k x}$$

holds pointwise and both sums converge absolutely.

Exercise 2:
Prove the following statements.

1. For $f \in C^p(\mathbb{T})$ it holds that

$$c_k\left(f^{(p)}\right) = (2\pi \mathrm{i} k)^p \, c_k(f), \qquad k \in \mathbb{Z}.$$

2. For real-valued $f \in L^2(\mathbb{T})$ the Fourier-coefficients satisfy

$$\overline{c_{-k}(f)} = c_k(f), \qquad k \in \mathbb{Z}.$$

3. For even $f \in L^2(\mathbb{T})$, i.e. $f(x) = f(-x)$, the Fourier-coefficients satisfy

$$c_{-k}(f) = c_k(f), \qquad k \in \mathbb{Z}.$$

4. The normalised and unshifted Fourier-matrix is unitary, i.e.,

$$\tilde{\mathbf{F}}_N^{\mathrm{H}} \tilde{\mathbf{F}}_N = \mathbf{I}_N \qquad \text{for } \tilde{\mathbf{F}}_N := \frac{1}{\sqrt{N}} \left(\mathrm{e}^{-2\pi \mathrm{i} k j / N}\right)_{j,k=0}^{N-1}.$$

Exercise 3:
Prove the following convolution theorems.

1. For $f, g \in L^2(\mathbb{T})$, $c_k(f) = \langle f, \mathrm{e}^{2\pi \mathrm{i} k \cdot}\rangle,\ c_k(g) = \langle g, \mathrm{e}^{2\pi \mathrm{i} k \cdot}\rangle,\ k \in \mathbb{Z}$ and

$$(f *_p g)(x) := \int_{\mathbb{T}} f(t)\, g(x-t)\, \mathrm{d}t$$

holds that

$$c_k(f *_p g) = c_k(f)\, c_k(g), \qquad k \in \mathbb{Z}.$$

2. For $f, g, fg \in L^2(\mathbb{T})$, $\mathbf{c}(f) = \left(\langle f, \mathrm{e}^{2\pi \mathrm{i} k \cdot}\rangle\right)_{k\in\mathbb{Z}},\ \mathbf{c}(g) = \left(\langle g, \mathrm{e}^{2\pi \mathrm{i} k \cdot}\rangle\right)_{k\in\mathbb{Z}}$ and

$$\left(\mathbf{c}(f) *_d \mathbf{c}(g)\right)_k := \sum_{l\in\mathbb{Z}} c_l(f)\, c_{k-l}(g)$$

holds that

$$c_k(fg) = \left(\mathbf{c}(f) *_d \mathbf{c}(g)\right)_k, \qquad k \in \mathbb{Z}.$$

3. For $\mathbf{f}, \mathbf{g} \in \mathbb{C}^n$, $\hat{\mathbf{f}} = \mathbf{F}_n \mathbf{f},\ \hat{\mathbf{g}} = \mathbf{F}_n \mathbf{g}$ and

$$(\mathbf{f} *_c \mathbf{g})_l := \sum_{j=-n/2}^{n/2-1} f_j g_{l-j},$$

where the index $l - j$ wraps around periodically, holds that

$$\left(\mathbf{F}_n (\mathbf{f} *_c \mathbf{g})\right)_k = \hat{f}_k \hat{g}_k.$$

Exercise 4:
Write a MATLAB-function `ndft(x,f_hat)` that computes the vector `f` with values

$$f(x_j) = \sum_{k=-N/2}^{N/2-1} \hat{f}_k\, \mathrm{e}^{2\pi \mathrm{i} k x_j}, \qquad j = 0, \dots, M-1.$$

Recast in matrix vector notation, compute

$$\mathbf{f} = \mathbf{A}\hat{\mathbf{f}}, \qquad \mathbf{A} := \left(\mathrm{e}^{-2\pi \mathrm{i} k x_j}\right)_{j=0,\dots,M-1;\, k=-N/2,\dots,N/2-1}.$$

Exercise 5:
Write the circulant convolution $\mathbf{f} *_c \mathbf{g}$ as matrix vector product $\mathbf{G}\mathbf{f}$. What are the singular values of $\mathbf{G}$.

Write your own MATLAB-function `fast_toeplitz(c,r,x)` for the fast computation of the matrix vector product `toeplitz(c,r)*x`.

Hint: Construct a circulant matrix `C` into which the Toeplitz matrix is embedded. Use the MATLAB functions `fft`, `ifft` for the matrix vector product `C*[x; zeros(size(x))]`. Do not use `C` explicitly.

Exercise 6:
Write a MATLAB function `f=taylor_nfft(x,f_hat,sigma,m)` that computes an approximation to the `ndft` by evaluating f and its derivatives on the oversampled grid $\frac{l}{\sigma N}$ and then using an mth order Taylor expansion of $f(x_j)$ about the nearest neighbour on this grid. Compare the computed values with the result of an `ndft` call. (Hint: The evaluation at the oversampled grid can be done by `fft` and `fftshift` calls. However, you might use the (slower) `ndft` for this step initially.)

Exercise 7:
Which of the following images and spectra match?