A Problem in Helioseismology:

determining the interior of the sun from its acoustic spectrum.

William Rundell
Texas A&M University

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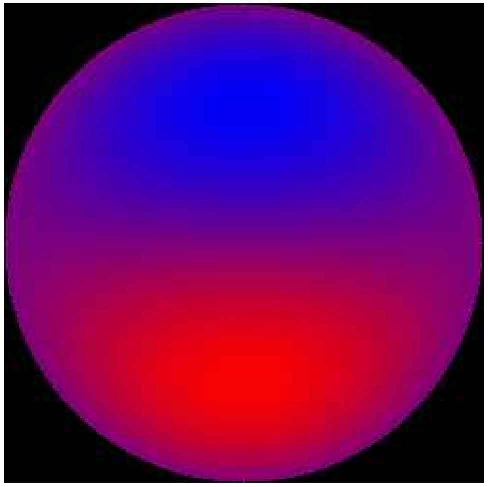
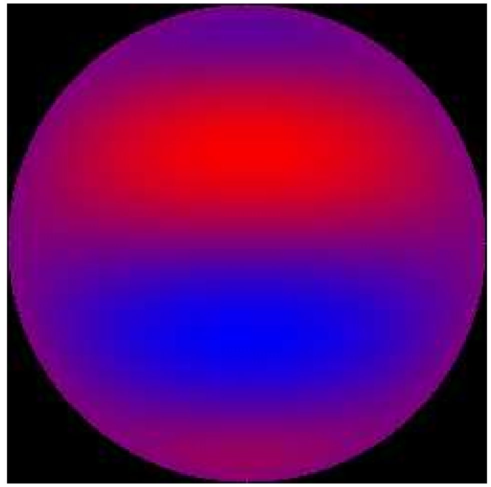


$\ell = 1$, $m = 0$ (equatorial)



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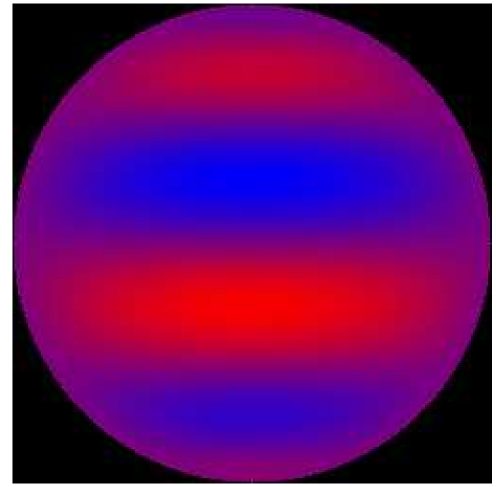
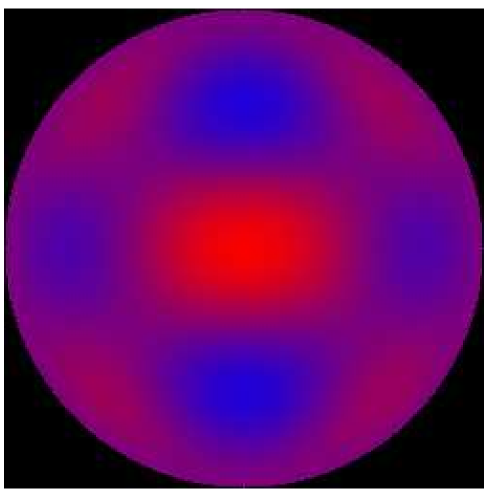


$\ell = 5$, $m = 0$ (equatorial)



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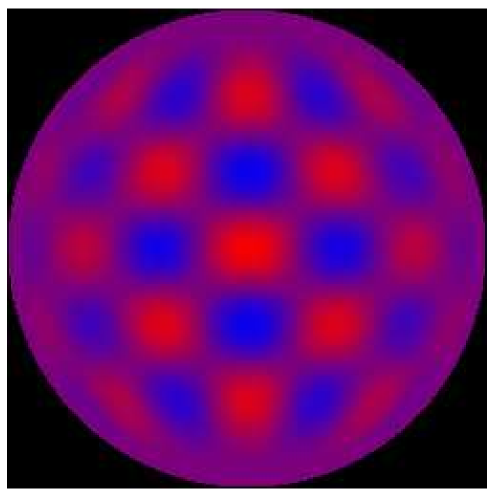
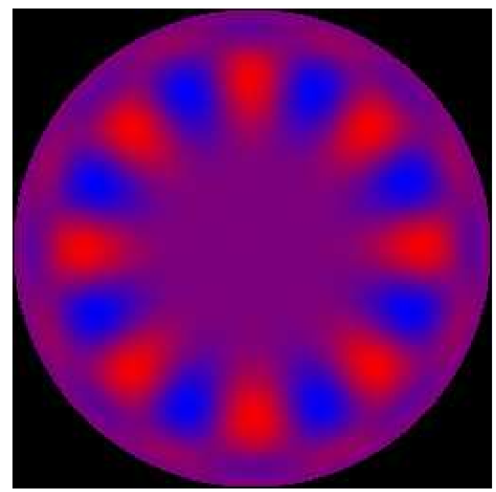


$\ell = 12$, $m = 8$ (equatorial)



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The acoustic vibrational amplitude is $u(r, \theta, \phi; t) = \psi(r; t) Y_\ell^m(\theta, \phi)$. separating the variables in the $\mathbb{R}^3$ Laplacian gives a radial equation of the form

$$\psi'' + \left(\frac{\lambda}{c^2(r)} - Q(r, \ell, \lambda) - \frac{\ell(\ell+1)}{r^2} \right) \psi = 0$$

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Two functions of importance; the propagation speed $c(r)$ and the density $\rho(r)$.

$$Q(r,\ell,\lambda) = Q_1(r) + \frac{Q_2(r)}{\lambda}$$

Q_1 depends only on $\rho(r)$, Q_2 depends on both $c(r)$ and $\rho(r)$ as well as ℓ.

$$Q_1(r) = \frac{2H'(r) - 1}{4H^2(r)} \qquad H = \frac{\rho(r)}{\rho'(r)}$$

$$Q_2(r,\ell) = \frac{\ell(\ell+1)}{r^2}N^2(r), \qquad N = g(1/H - g/c^2), \qquad g = \frac{4\pi G}{3r^2}\int_0^r \rho(s)\,ds$$

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The most realistic boundary condition is of the form $\psi'(1) - h\psi(1) = 0$ where the parameter h is also to be determined.

There are enormous amounts of data: GONG (**G**lobal **O**scillation **N**etwork **G**roup) data consists of $\lambda_{\ell,n}$'s with ℓ from 0 to 1,000 and n from 1 to about 50-100. Accuracy is very good in an absolute scale, but poor from a usable standpoint. No other spectral information is readily available.

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If we use the Liouville transform on

$$-\psi'' + \left(Q(r,\lambda) + \frac{\ell(\ell+1)}{r^2}\right)\psi = \frac{\lambda}{c^2(r)}\psi$$

we can remove the term containing c, modifying the Q, but also modifying the singular term. If we ignore this, then a canonical form might be

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If the terms in colour are removed, this is a standard Sturm-Liouville problem:

> $q = Q_1$ is uniquely determined by the eigenvalues $\{\lambda_{0,n}\}_1^\infty$, together with a second sequence: norming constants, end-point values, or a second spectral sequence corresponding to another boundary condition at $r = 1$.

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> The helioseismology application does not allow a change in the boundary values or the measurement of anything other than eigenvalue data.

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We can say nothing about the critical question of uniqueness for the full problem.

Break into two simpler problems - each containing only one of the coloured terms.

Optimal Method for the Regular Sturm Liouville Problem

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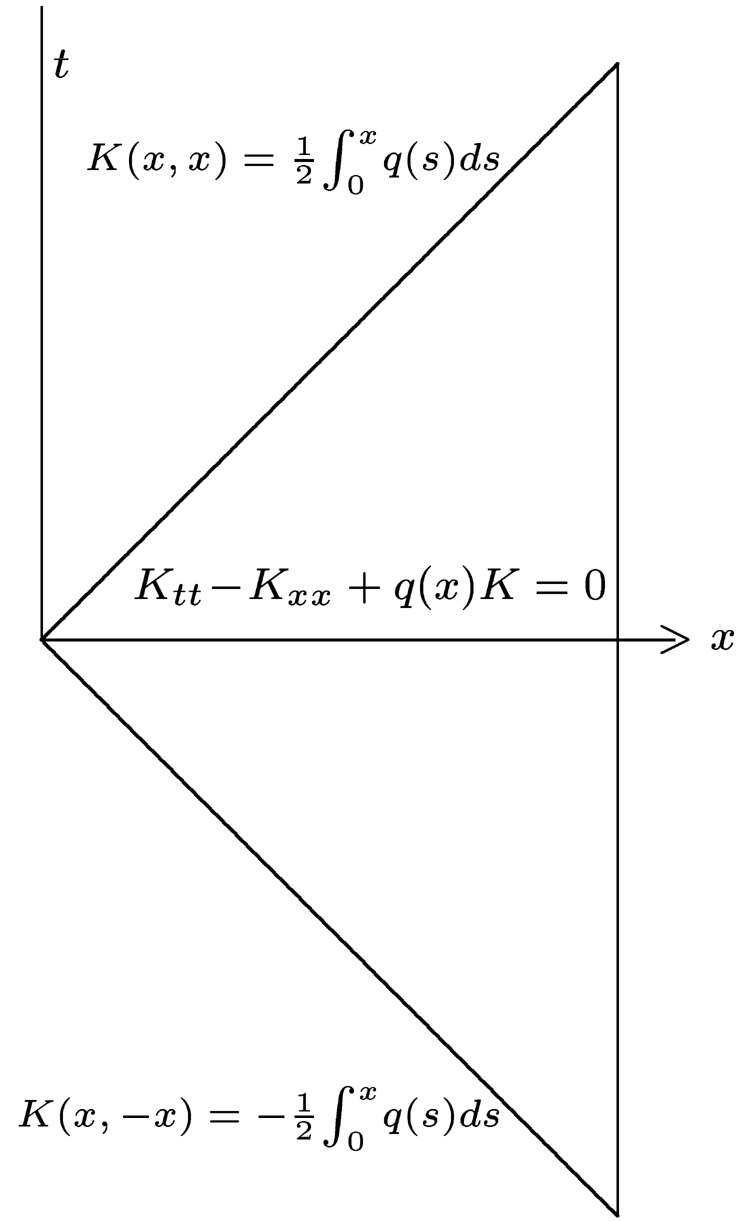

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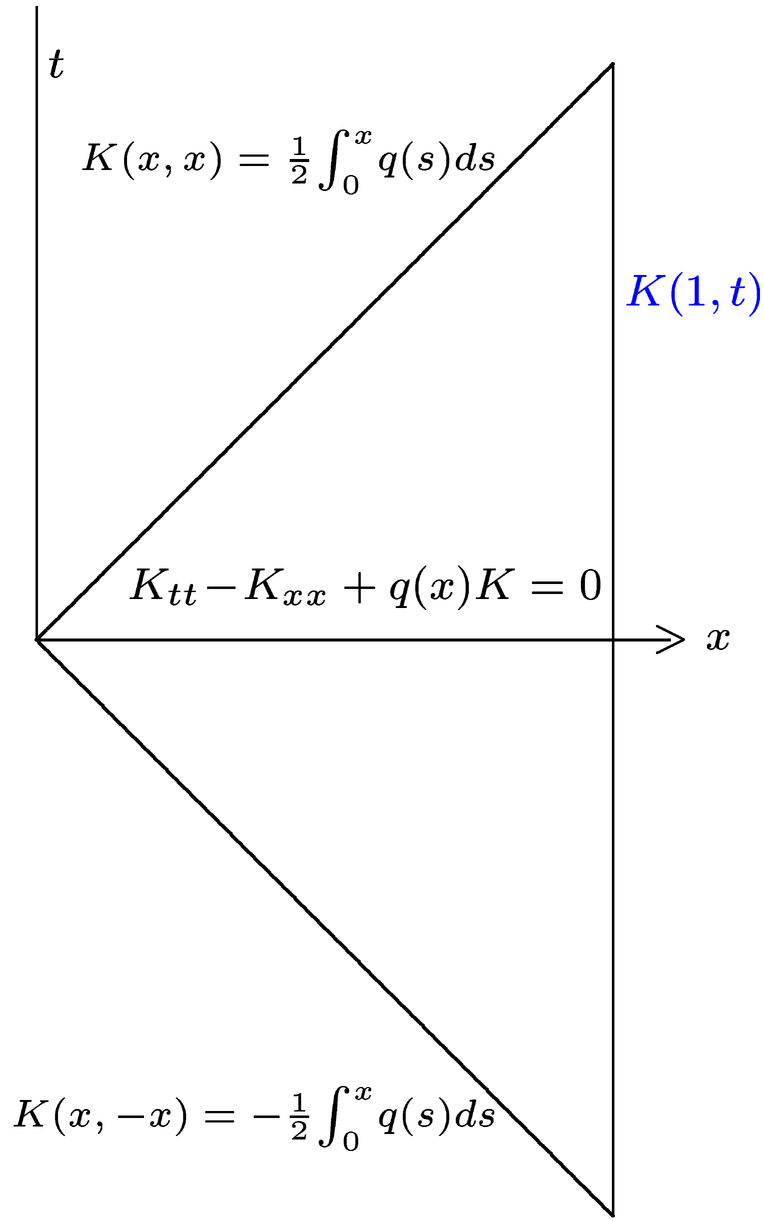
$\int_0^1 K(1,t)\sin(\sqrt{\lambda}t)\,dt = -\sin\sqrt{\lambda}$

This equation then gives $K(1,t)$.



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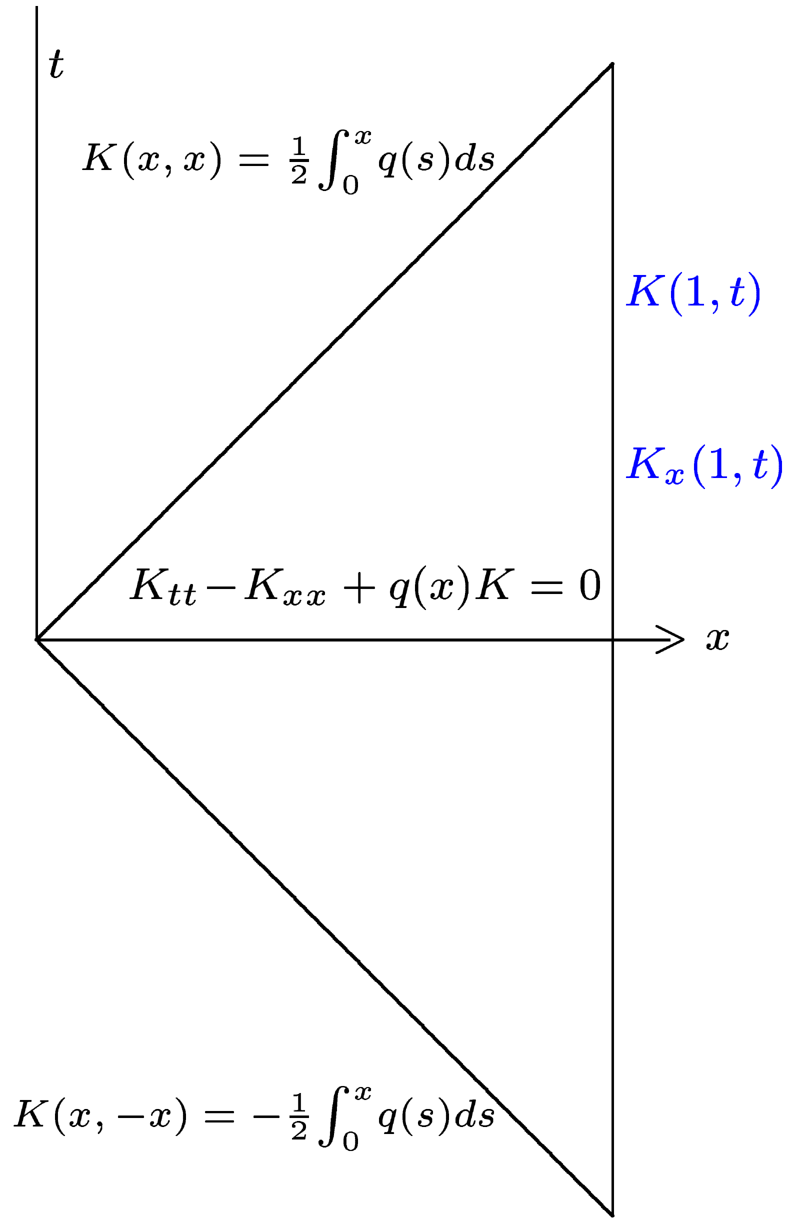

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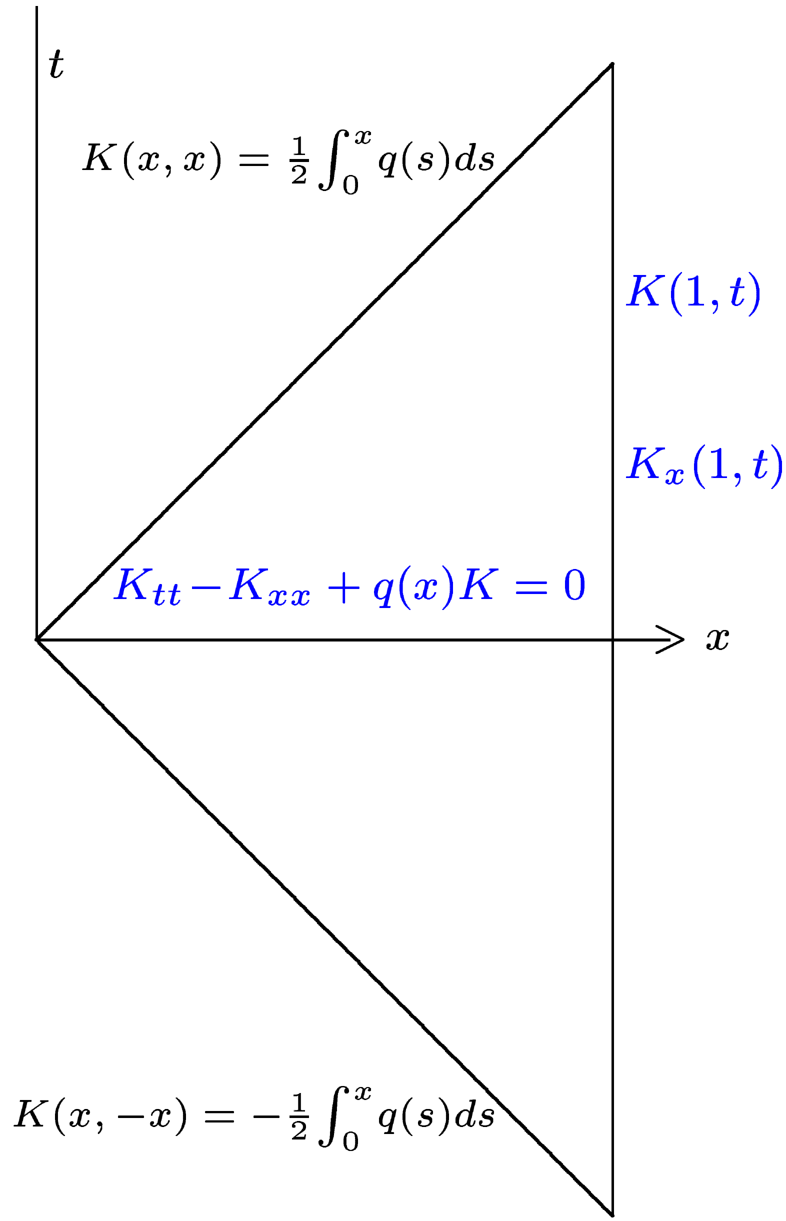

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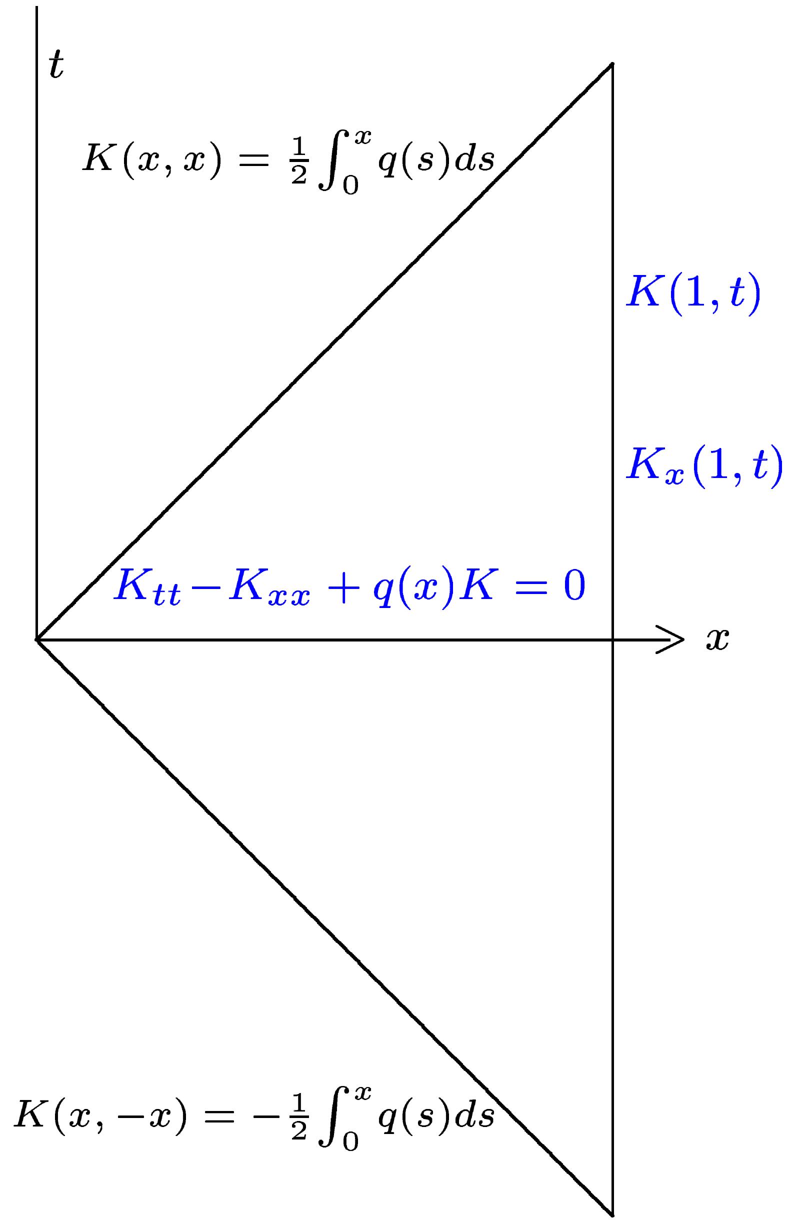
Iterate $q_{n+1} = K(x,x;q_n)$ to recover $q(x)$



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We consider the eigenvalue problem for $\ell = 0, 1, 2, \ldots$.

$$\begin{gathered} \psi'' + \left(\lambda - q(r) - \frac{\ell(\ell+1)}{r^2}\right)\psi = 0 \qquad 0 < r < 1 \\ \psi(1) = 0 \qquad \psi(r) = O(r) \quad r \to 0 \end{gathered} \tag{1}$$

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$\ell = 0$ is the classical Inverse Sturm-Liouville problem:

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The goal is to recover $q(r)$ from (some subset of) the spectral data $\{\sqrt{\lambda}_{\ell,n}\}$.

The eigenvalues have the following asymptotic values

$$\sqrt{\lambda}_{\ell,n} = (n+\tfrac{\ell}{2})\pi + \frac{\int_0^1 q(x)\,dx - \ell(\ell+1)}{(2n+\ell)\pi} + \beta_{\ell,n}, \qquad \sum_{n=1}^{\infty} n\beta_{\ell,n}^2 < \infty.$$

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It is always instructive to look at the simplest case.

Case 1: Including only the singular term $\ell(\ell+1)/r^2$

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For fixed ℓ, (1) has a countable sequence of eigenvalues, $\lambda_{\ell,n}$, $n = 1, 2, \ldots$.

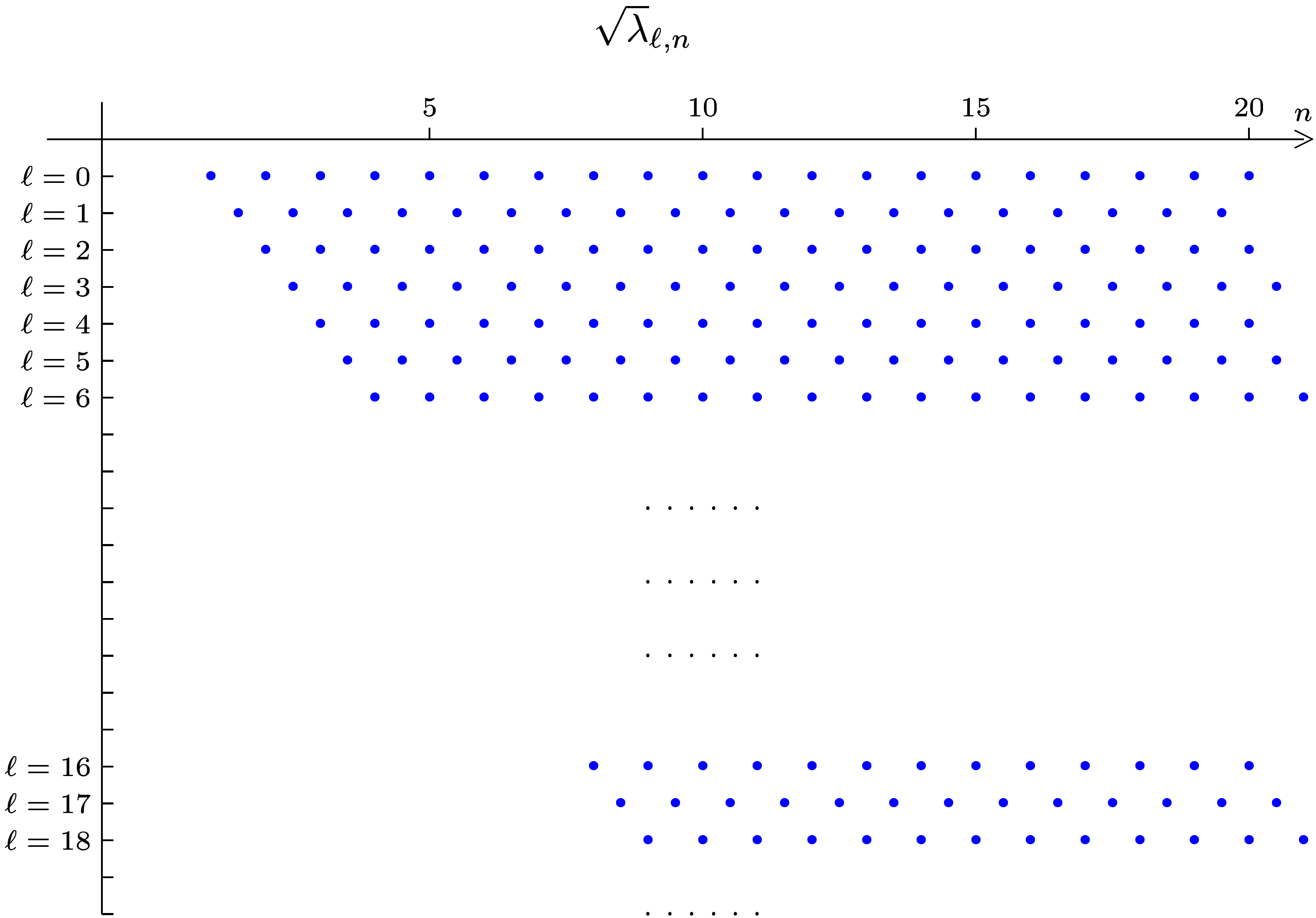
If $q(r) = 0$ and we take Dirichlet conditions at $r = 1$, then the eigenfunctions are

$$\psi(r) = r^{\ell+1} j_\ell(\sqrt{\lambda} r)$$

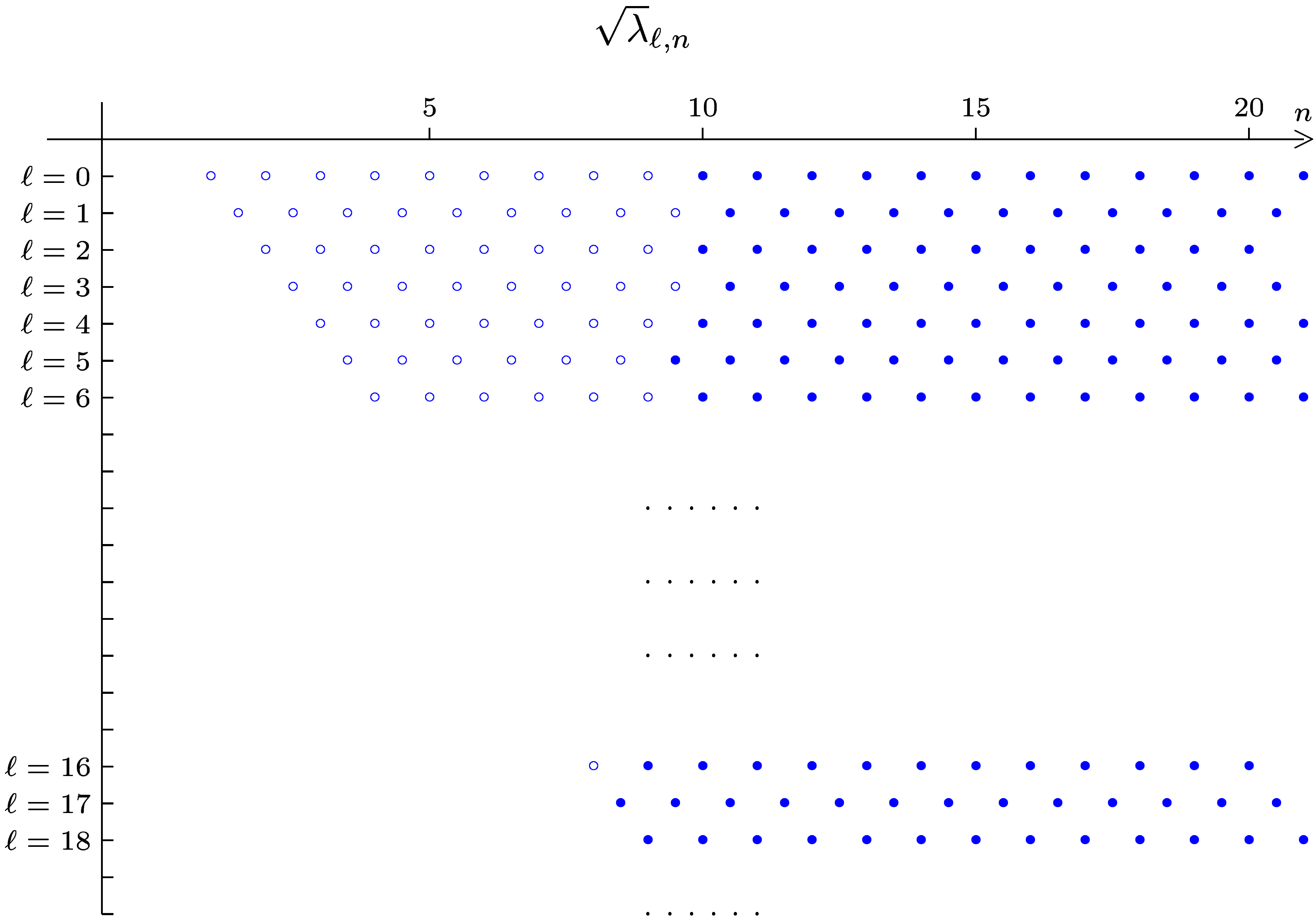
where j_ℓ is the spherical Bessel function.

The eigenvalues are the positive roots of $j_\ell(\sqrt{\lambda}) = 0$.

For nonzero q we expect the eigenvalues and eigenfunctions to have similar properties - at least for a sufficiently small $q(r)$.



$\sqrt{\lambda}_{\ell,n}$
5
10
15
20
n
$\ell = 0$
$\ell = 1$
$\ell = 2$
$\ell = 3$
$\ell = 4$
$\ell = 5$
$\ell = 6$
$\ell = 16$
$\ell = 17$
$\ell = 18$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5
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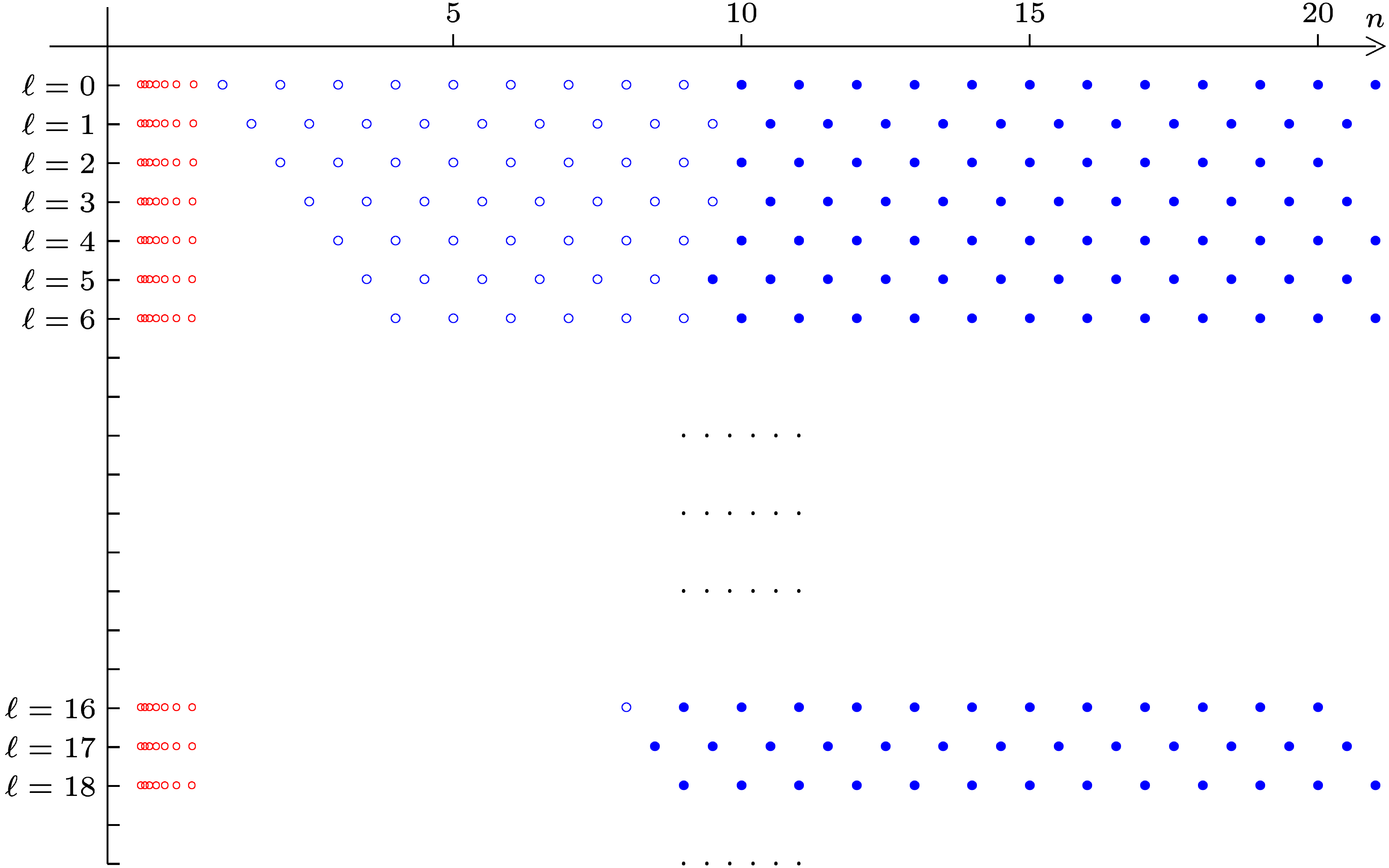
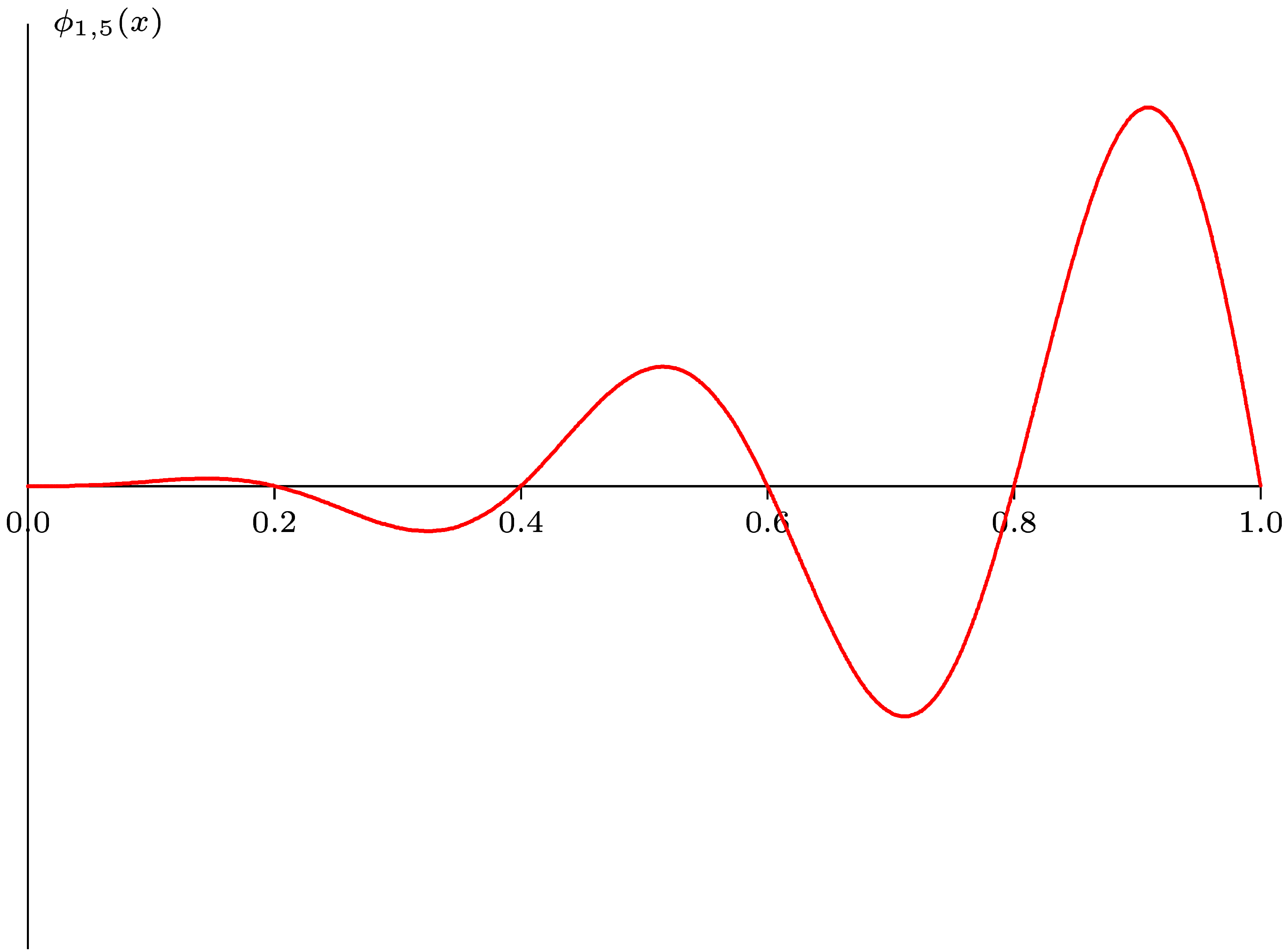
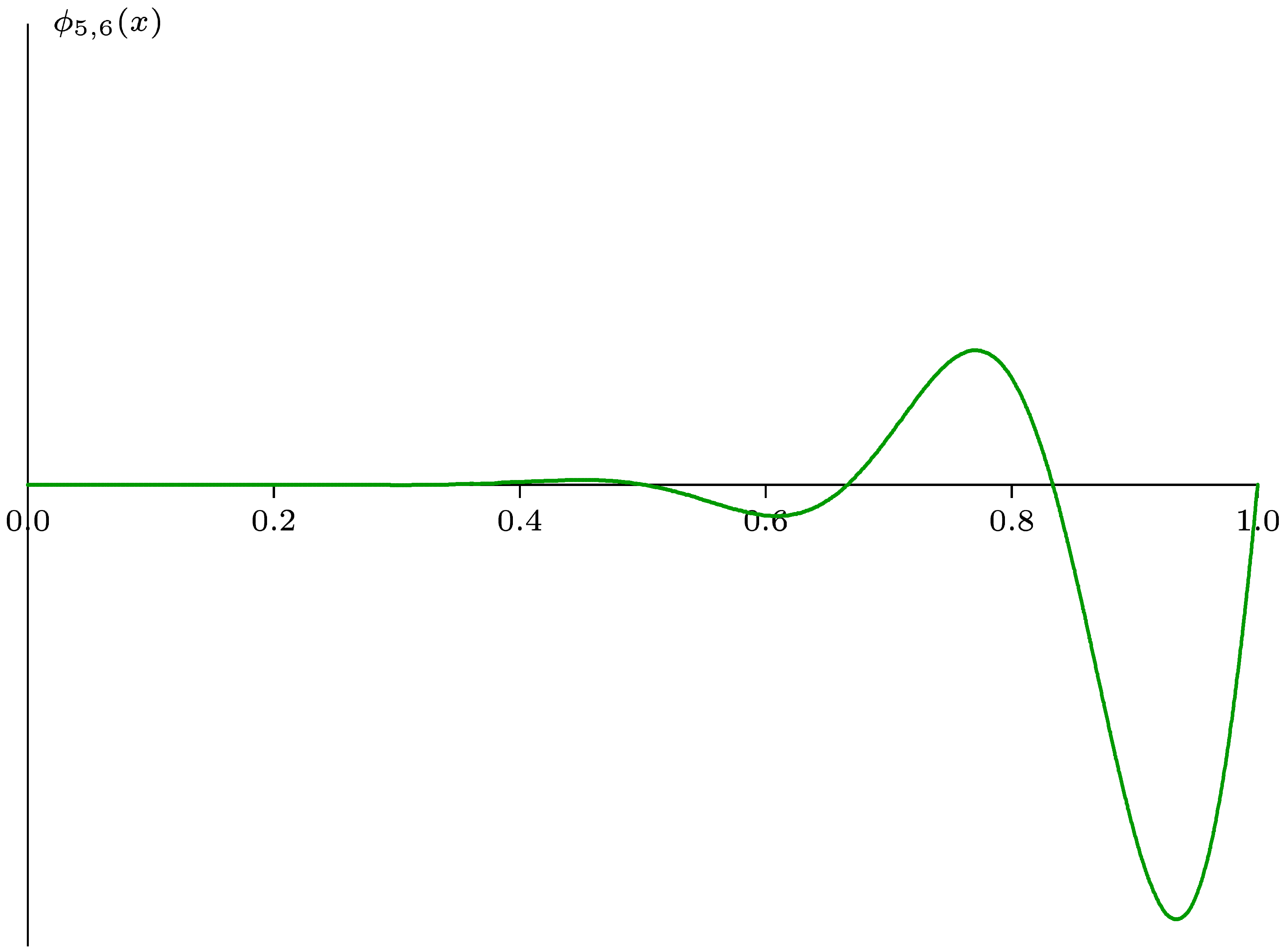



$\phi_{1,5}(x)$
0.0
0.2
0.4
0.6
0.8
1.0


$\phi_{5,6}(x)$
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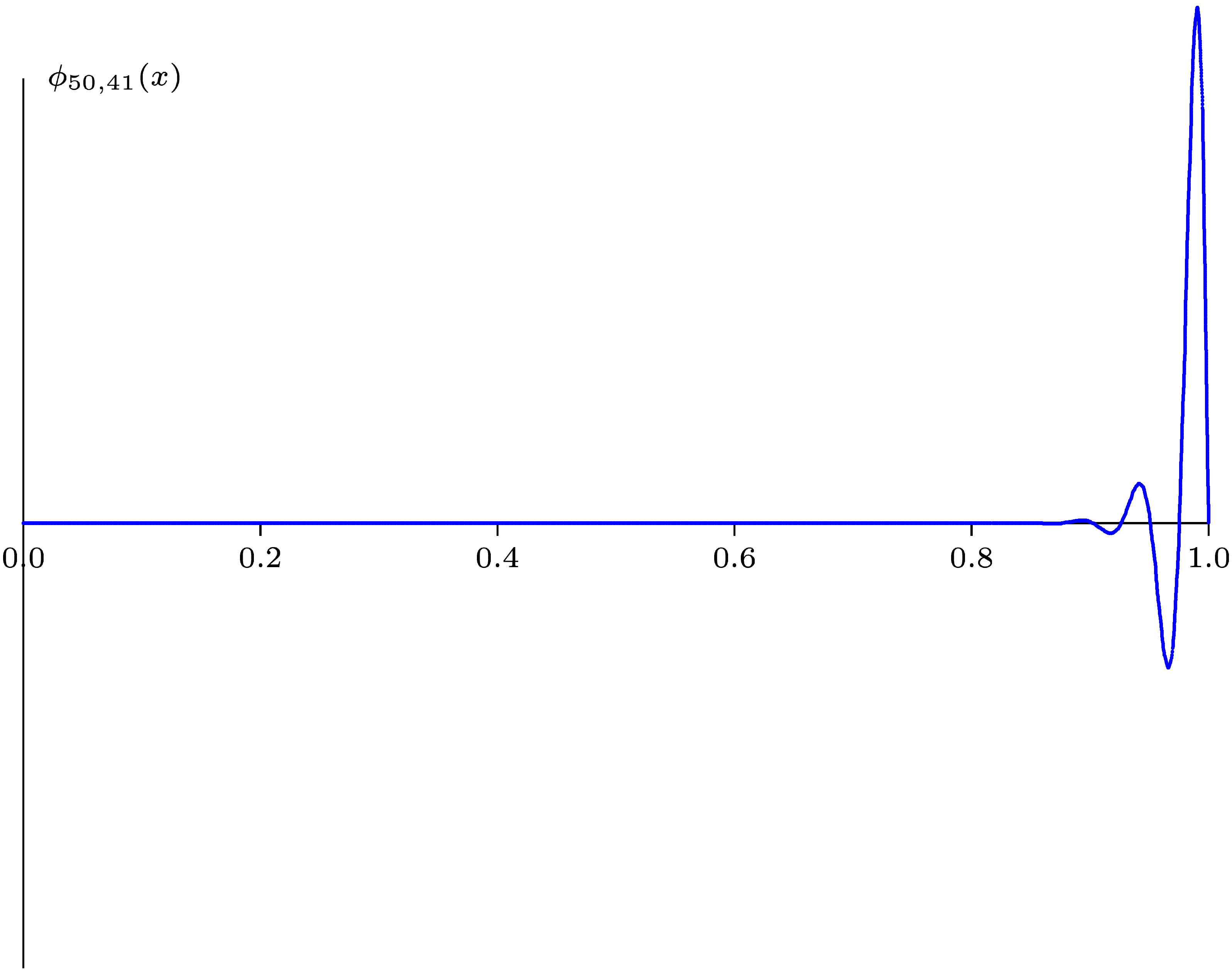
1.0


$\phi_{50,41}(x)$
0.0
0.2
0.4
0.6
0.8
1.0

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Carlson and Shubin showed that the set of potentials sharing the same two spectral sequences is locally of finite dimension provided that $\ell_1 - \ell_2$ is an odd integer.

There are positive answers (and reconstructions) for cases with small ℓ: for example, $\ell = \{0, 1\}$, $\ell = \{0, 2\}$, $\ell = \{1, 2\}$, $\ldots$ (Rundell, Sacks).

The forward map.

We formulate the inverse spectral problem as a nonlinear operator equation; for each value of $\lambda \in \Lambda$ define u to be the solution of

$$u'' + \left(\lambda - q(r) - \frac{\ell(\ell+1)}{r^2}\right)u = 0 \qquad 0 < r < 1$$

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The question: if $\Lambda = \left\{\{\lambda_{\ell_1,n}\}_{n=1}^{\infty}, \{\lambda_{\ell_2,n}\}_{n=1}^{\infty}\right\}$ for some ℓ_1, ℓ_2, does the equation

$$F_\Lambda(q) = 0 \tag{2}$$

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It is natural to attempt to solve (2) by some version of Newton's method

$$q_{n+1} = q_n - D_q F_\Lambda^{-1}(q_n) F_\Lambda(q_n) \tag{3}$$

and this requires some insight into the structure of the linearized map $D_q F_\Lambda(q)$.

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We must show local injectivity of F', that is if $D_q F_\Lambda(q)\zeta = 0$, then $\zeta = 0$.

$$\int_0^1 \psi_\ell^2(x)\zeta(x)\,dx = 0 \quad \text{for} \quad \ell = \ell_1, \ell_2 \qquad \Rightarrow \quad \zeta = 0\,??$$

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The mean value $\int_0^1 q(x)\,dx$ is uniquely determined by the asymptotics of the eigenvalues, for any fixed ℓ.

$$\sqrt{\lambda}_{\ell,n} = (n+\frac{\ell}{2})\pi + \frac{\int_0^1 q(x)\,dx - \ell(\ell+1)}{(2n+\ell)\pi} + \beta_{\ell,n},$$

$\Rightarrow$ by a preliminary calculation we may always assume that $\int_0^1 q(x)\,dx = 0$.

Hence need only consider those $\zeta(=\delta q)$ with $\int_0^1 \zeta = 0$.

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$$D_q F_\Lambda(0)\zeta = C\int_0^1 \sin^2(\sqrt{\lambda^0_{\ell,n}}x)\,\zeta(x)\,dx\Big|_{n\in\Lambda}$$

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Thus $D_q F_{\Lambda_o}(0)\zeta = 0$ implies ζ is odd.

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Then S_ℓ is bounded and one to one on $L^2(0,1)$, The function $\{x^{2\ell}\}$ is the only element in the nullspace of S_ℓ^* and $\psi_\ell^2 = -S_\ell^*[\psi_{\ell-1}^2]$.

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This leads to

Lemma 4. **If** $\sqrt{\lambda} \approx n\pi$, $n = 1,\ 2,\ \ldots$ *then $F'[0]\zeta = 0$ implies*

$$T_\ell[\zeta] = 0 \quad \text{for} \quad \ell = \ell_1,\ \ell_2$$

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where $\chi_e(x) = \chi_e(1-x)$, $\chi_o(x) = -\chi_o(1-x)$, ($\epsilon_i \in R$).

$$\mathbf{T}_1[f] = f(x) - 4x\int_x^1 \frac{f(s)}{s^2}\, ds \qquad \mathbf{T}_2[f] = -f(x) - 12x\int_x^1 \frac{f(t)}{t^2}\, dt + 24x^3\int_x^1 \frac{f(t)}{t^4}\, dt$$

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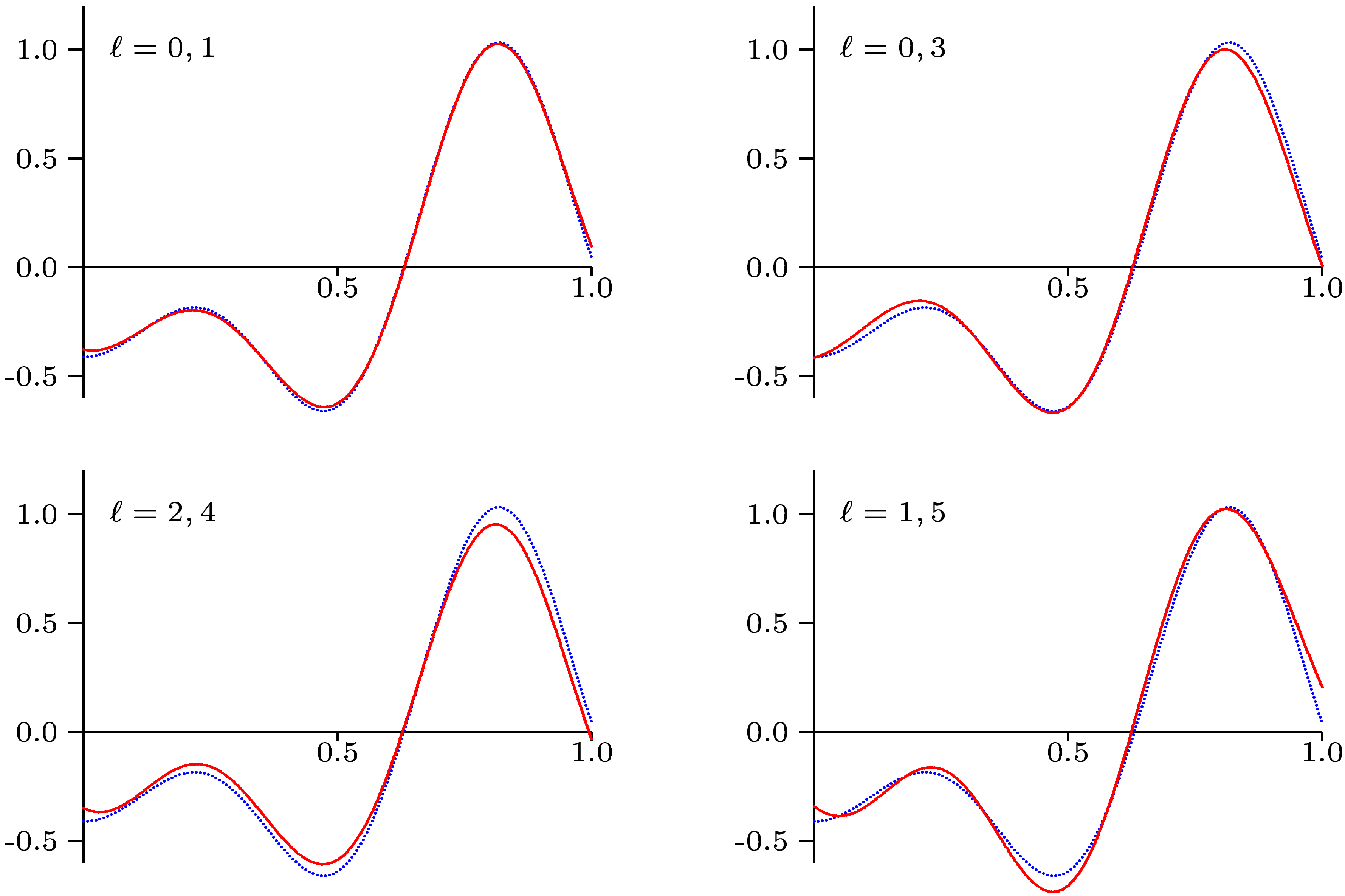
We have accomplished this for several pairs of ℓ values - $(0, 1), (1, 2), (0, 2), (1, 3)$ and can show that the restriction of $\ell_1 - \ell_2$ odd can be removed.

Reconstructions with 5% error in $\beta_{n,\ell}$



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- Not all spectral sequences $\{\lambda_{\ell,n}\}_{n=1}^{\infty}$ for different ℓ values carry the same information content about q (we would prefer small ℓ). It is certainly the case that the error in the spectra also varies with ℓ. If we use more data than is necessary, how do take all of this into account?

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Since we no longer have different ℓ values, we generate two spectra by changing the boundary conditions at $r = 1$: $\{\lambda_n\}$ corresponding to $\psi(1) = 0$ and $\{\tilde{\lambda}_n\}$ to $\psi'(1) = 0$.

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We can write (4) in the form of a quadratic eigenvalue problem:

$$(\lambda^2\mathcal{A} + \lambda\mathcal{B} + \mathcal{C})\psi = 0$$

where $\mathcal{A} = I$, $\mathcal{B} = -D^2 + Q_1(x)I$, $\mathcal{C} = -Q_2(x)I$. These are all self-adjoint, and provided $Q_1(x) \geq 0$, $Q_2(x) < 0$, are also positive operators. Under these conditions the spectrum is real, positive, and consists of two sequences $\{\mu_n\}$, $\{\eta_n\}$ with $\mu_1 < \mu_2 < \mu_3 < \ \ldots \ , \mu_n \to \infty$ and $\mu_1 > \eta_1 > \eta_2 > \eta_3 > \ \ldots \ , \ \eta_n \to 0$.

For our particular operators these sequences have the asymptotic form

$$\mu_n = n\pi + \frac{\int_0^1 Q_1(s)\,ds}{2n} + O(n^{-2}) \qquad \eta_n = \frac{L}{n\pi} + O(n^{-2}), \quad L := \int_0^1 \sqrt{Q_2}(s)\,ds$$

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Suppose we are given the two pairs of sequences $\Lambda := \big\{ \{\mu_n, \eta_n\}_{n=1}^N, \{\tilde{\mu}_n, \tilde{\eta}_n\}_{n=1}^N \big\}$ arising from Dirichlet and Neumann conditions at $r = 1$.

Represent Q_1, Q_2 as finite term Fourier series $\quad Q(r) = a_0 + \sum_1^{2N} a_n \cos(n\pi r)$, and for a given set Λ, define the map $F_\Lambda : L^2[0,1] \times L^2[0,1] \to \mathbb{R}^{4N}$ by

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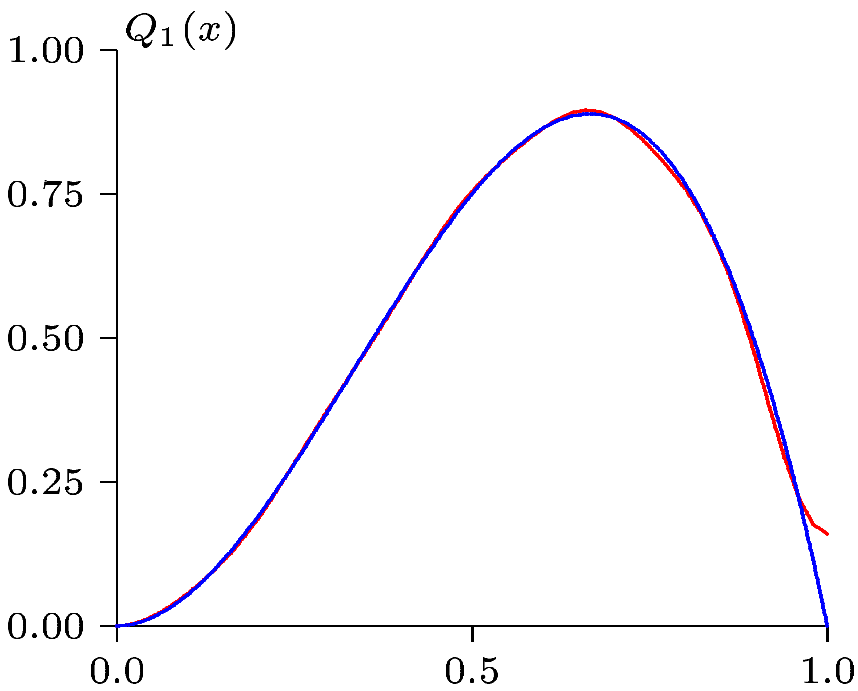
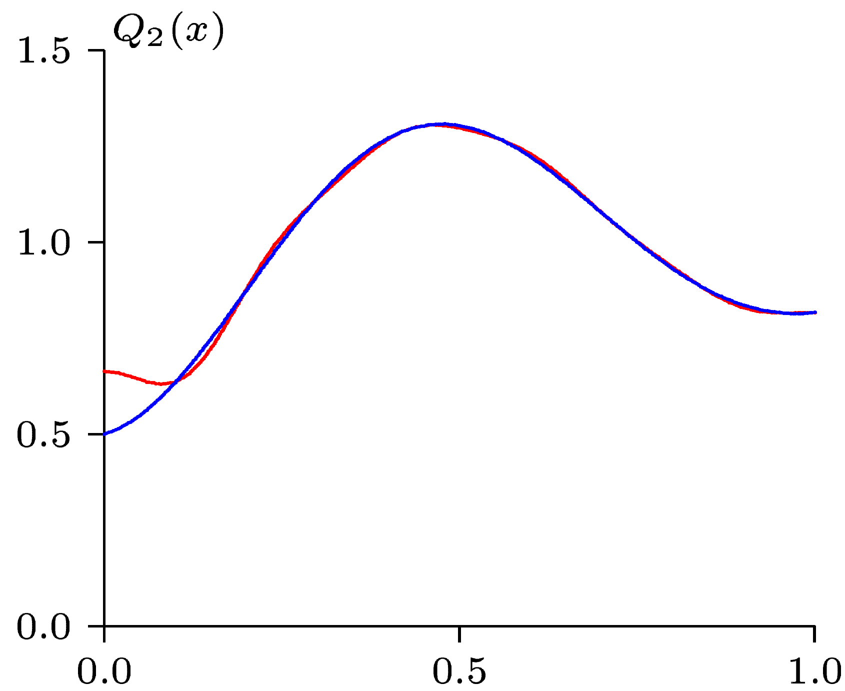
Proof: Use the asymptotic expansions to show that the block matrix representation of F' is diagonally dominant.



$Q_1(x)$
1.00
0.75
0.50
0.25
0.00
0.0
0.5
1.0



$Q_2(x)$
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1.0
0.5
0.0
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