On the relation between constraint regularization and level-set methods

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Overview of the Talk

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Work in cooperation with Prof. Dr. O. Scherzer (Univ. Innsbruck)

§ 1 The Inverse Problem

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- Determination of an Unknown set from indirect measurements of it's characteristic function.

- Abstract Model:

— $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$; $D \subset\subset \Omega$;

— $u = \begin{cases} u_{int}, & x \in D \\ u_{ext}, & x \notin D \end{cases}$.

— Find the domain D in the equation $F(u) = g$.
g represents the problem data;
D (or u) represents the (unknown) model parameters.

Applications:

- Inverse Scattering for an obstacle (Colton & Kress, Springer Book 1992):

— Operator F represents the map to the far field pattern from a scatterer D, for a given set of incident waves.

- Mine Detection (Friedman, SIAM J Appl. Math., 1987):

— u_{int} is the conductivity of the mine;
— u_{ext} is the conductivity of the soil;
— unknown: domain $D \subset\subset \Omega$.

- Deblurring (deconvolution):

— $F : L^2(\Omega) \to L^2(\Omega)$ is the operator

$$F(u) = \int_\Omega k(x, y)u(y)\, dy.$$

(F compact and bounded on $L^2(\Omega)$ if $k \in L^2(\Omega \times \Omega)$)
— $u = \chi(D), \; D \subset\subset \Omega$;
— $u_{int} = 1, \quad u_{ext} = 0$;
— Deconvolution problems: $k(x, y) = \mathcal{K}(x - y)$, for some $L^1(\Omega)$ kernel $\mathcal{K}$.

- Inverse potential problem (Hettlich & Rundell, I.P. 1996):

— $D \subset\subset \Omega$ is an object to be reconstructed in

$$\Delta U = \chi(D), \ \Omega \quad \text{with} \quad U = 0, \ \partial\Omega.$$

— $F_1 : L^2(\Omega) \to H^2(\Omega)$ is the operator

$$f \mapsto \Delta^{-1} f, \quad \text{with homogeneous Dirichlet data.}$$

— $F_2 : L^2(\Omega) \to H^{1/2}(\partial\Omega)$ is the operator

$$f \mapsto (\Delta^{-1} f)_\nu|_{\partial\Omega}, \quad \text{with homogeneous Dirichlet data.}$$

Remark: The operator F is the forward operator, i.e. a map from the model parameter space to the data space.

§ 2 Level-Set Approach (Osher & Sethian; Santosa)

- $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

- $D \subset\subset \Omega$.

- ∂D described by a function $\phi : \Omega \to \Omega$: $\partial D = \{x \, in\Omega;\ \phi(x) = 0\}$.

- Level-set approach:

— generate a sequence of functions $\phi_k(x)$ such that

$$D_k \to D \quad \text{where} \quad \partial D_k = \{x \, in\Omega;\ \phi_k(x) = 0\}.$$

— k may be a continuous parameter.
— Level-set representation of the function u:

$$u(x) \;=\; \begin{cases} u_{int}, & \{x \, in\Omega;\ \phi_k(x) < 0\} \\ u_{ext}, & \{x \, in\Omega;\ \phi_k(x) > 0\} \end{cases}$$

- Characteristics:

— No *a priori* assumptions on the topology of D is required.
— The dependencd of u on ϕ is nonlinear.
(even if F is linear, the inverse problem becomes nonlinear)

§ 3 Level-Set Evolution Equation

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- Gateaux derivatives:

— Calculate the variation of u caused by a variation of ϕ;

— Notation:

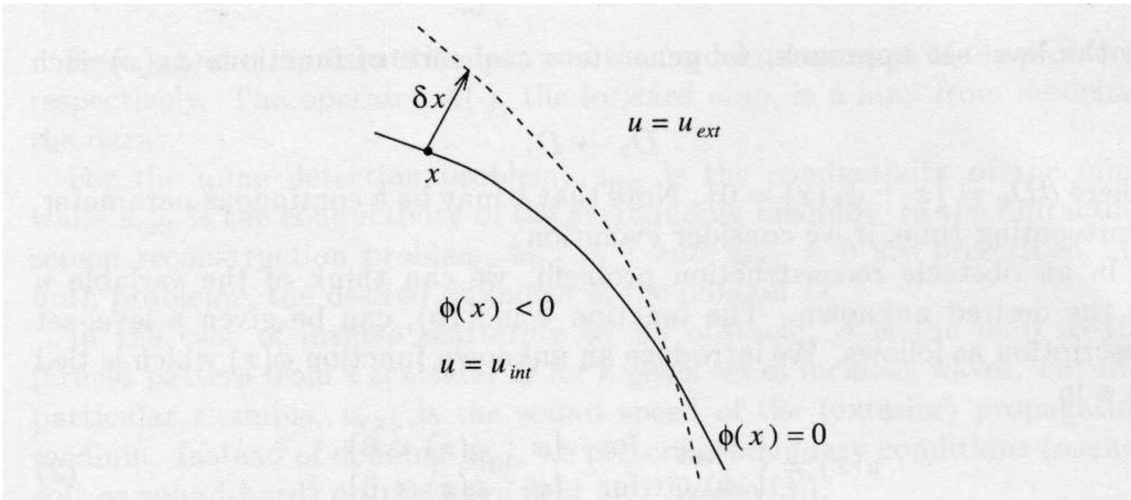
x point on the surface ∂D $\{x \in \Omega;\ \phi(x) = 0\}$;

$\phi(x)$ is perturbed by a small variation $\delta\phi(x)$;

δx is the resulting variation of the point x;

D' is the new region originated from D after the perturbation;

δu variation of the function u (to be computed);



- Formal variation of the equation $\phi(x) = 0$:

$$\delta\phi + \nabla\phi \cdot \delta x = 0.$$

- Variation of u:

$$\langle \delta u, f \rangle = \int_\Omega \delta u(x) f(x)\, dx = \int_{D\Delta D'} \delta u(x) f(x)\, dx$$

for a given test function f. $\Bigg($the value of δu is $\delta u = \begin{cases} u_{int} - u_{ext} \\ u_{ext} - u_{int} \end{cases}\Bigg)$

$$\langle \delta u, f \rangle = \int_{\partial D} (u_{int} - u_{ext}) \delta x \frac{\nabla\phi(x)}{|\nabla\phi(x)|} f(x) ds(x)$$

(normal vector: $n(x) = \nabla\phi(x)/|\nabla\phi(x)|$)

Therefore: $\delta u = (u_{int} - u_{ext}) \dfrac{\nabla\phi(x)}{|\nabla\phi(x)|} \cdot \delta x|_{x \in \partial D}$.

(the dot product induces the correct sign for δu)

- Derivation of an evolution equation for the level-set function:

— $\phi = \phi(x,t), \;\; x \in \Omega, \;\; t \geq 0$;

— $\partial D(t) = \{x \in \Omega; \; \phi(x,t) = 0\}$;

— Solve the inverse problem in least square sense:

$$J(u) \;:=\; \tfrac{1}{2}\|F(u) - g\|^2.$$

— The derivative $\partial\phi/\partial t$ should be chosen such that $J(u(t))$ is a decreasing function of t.

— Assumption: Each point $x \in \partial D(T)$ moves perpendicular to the surface, i.e.

$$\delta x \;=\; v(x,t)\frac{\nabla\phi}{|\nabla\phi|}.$$

($v(x,t)$ is the *velocity* of the surface $\partial D(T)$ at the point x and time t)

— Therefore: $\delta u \;=\; (u_{int} - u_{ext})\; v(x,t)|_{x\in\partial D(t)}$.

— Gateaux derivative of J at direction δu.

$$\begin{aligned} \delta J(u) \quad &= \quad \langle F'(u)^*(F(u) - g), \delta u \rangle \\ & \quad\quad \int_{\partial D(t)} [F'(u)^*(F(u) - g)]\ (u_{int} - u_{ext})\ v(x,t) ds(x). \end{aligned}$$

(non restritive assumption: $u_{int} > u_{ext}$)

— Natural choice of $v(x,t)$: $\quad v(x,t) = -F'(u)^*(F(u) - g)|_{x \in \partial D(t)}$.
($\delta J(u) \leq 0$)

— Corresponding variation of ϕ:

$$\delta\phi \ = \ -\nabla\phi \cdot \delta x \ = \ -\nabla\phi \left(v(x,t) \frac{\nabla\phi}{|\nabla\phi|} \right) \ = \ -v(x,t)\,|\nabla\phi|.$$

— Extension of $v(x,t)$ from $\partial D(t)$ to Ω:

$$v(x,t) \ = \ -F'(u)^*(F(u) - g).$$

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— Variation of ϕ:

$$\delta\phi \;=\; [F'(u)^*(F(u)-g)]|\nabla\phi|.$$

— Evolution equation for level-set function:

$$\begin{cases} \dfrac{\partial\phi}{\partial t} \;=\; [F'(u)^*(F(u)-g)]|\nabla\phi| \\ \phi(x,0) \;=\; \phi_0(x) \end{cases}$$

(Hamilton–Jacobi equation: $\frac{\partial\phi}{\partial t} + v|\nabla\phi| \;=\; 0$)

— Properties of the evolution method:

a) $J(u(t))$ is non increasing;

b) If $\bar{u}$ is a solution of $F(u) = g$ and $\phi(x,t)$ corresponds to u for some $\tau > 0$, then $\partial\phi/\partial t(x,\tau) \;=\; 0$.

c) The evolution corresponds to the steepest descent method for the residual

$$J(u) \;=\; \tfrac{1}{2}\|F(u)-g\|^2.$$

§ 4 Constraint Regularisation Approach

The general context is to solve the constraint ill-posed operator equation:

$$F(u) = y \, , \tag{1}$$

where u is in the admissible class

$$U := \{u : u = P(\phi) \text{ and } \phi \in \mathcal{D}(P)\} \, .$$

The constraint equation can be formulated as an unconstrained equation

$$F(P(\phi)) = y \, . \tag{2}$$

Tikhonov regularization for solving the unconstrained equation (1):
Approximate the solution of (1) by the minimizer u_α of the functional

$$\|F(u) - y\|^2 + \alpha \|u - u_*\|^2 \, .$$

If F is differentiable, then

$$F'(u_\alpha)^*(F(u_\alpha) - y) + \alpha(u_\alpha - u_*) = 0\,, \tag{3}$$

($F'(u_\alpha)^*$ denotes the adjoint $\cdot^*$ of the derivative of F at u_α)

Eq. (3) is the optimality condition for a minimizer of the Tikhonov functional.

Using the formal setting $\Delta t := 1/\alpha$, $u(\Delta t) := u_\alpha$, and $u(0) := u_*$ we find:

$$F'(u(\Delta t))^*(F(u(\Delta t)) - y) + \frac{u(\Delta t) - u(0)}{\Delta t} = 0\ .$$

Thus $u_\alpha = u(\Delta t)$ can be considered as the solution of one implicit time step with step-length $\Delta t = \frac{1}{\alpha}$ for solving

$$\partial u/\partial t = -F'(u)^*(F(u) - y). \tag{4}$$

(corresponds to the *asymptotic regularization method* as introduced by Tautenhahn)

Let $\Omega \subseteq \mathbb{R}^n$ be bounded with Lipschitz boundary $\partial\Omega$. Set

$$\mathcal{P} := \{u : u = \chi_{\tilde{\Omega}} : \tilde{\Omega} \subseteq \Omega\} \cap L^2(\Omega)\,,$$

The unconstrained inverse problem consists in solving (2) with

$$\begin{aligned} P : H^1(\Omega) &\to \mathcal{P}\,. \\ \phi &\mapsto \tfrac{1}{2} + \tfrac{1}{2}\mathrm{sgn}(\phi) =: \tfrac{1}{2} + \tfrac{1}{2}\begin{cases} 1 \text{ for } \phi \geq 0 \\ -1 \text{ for } \phi < 0 \end{cases} \end{aligned}$$

(Assumption: $F : L^2(\Omega) \to L^2(\Omega)$ is Fréchet-differentiable)

Tikhonov regularization for this problem consists in minimizing the functional

$$\int_\Omega (F(P(\phi)) - y)^2 + \alpha \int_\Omega \left((\phi - \phi_*)^2 + |\nabla(\phi - \phi_*)|^2\right)\,. \tag{5}$$

The functional in (5) may not attain a minimum!

We consider the "minimizer" ϕ_α as $\phi_\alpha = \lim_{\varepsilon\to 0+} \phi_{\varepsilon,\alpha}$,

where $\phi_{\varepsilon,\alpha}$ minimizes the functional

$$\int_\Omega (F(P_\varepsilon(\phi)) - y)^2 + \alpha \int_\Omega \left((\phi - \phi_*)^2 + |\nabla(\phi - \phi_*)|^2 \right) . \qquad (6)$$

We use

$$P_\varepsilon(t) := \begin{cases} 0 & \text{for} \quad t < -\varepsilon \,, \\ 1 + \frac{t}{\varepsilon} & \text{for} \quad t \in [-\varepsilon, 0] \,, \\ 1 & \text{for} \quad t > 0 \,, \end{cases}$$

for approximating P as $\varepsilon \to 0^+$.

In this case we have $P'(t) = \lim_{\varepsilon\to 0+} P'_\varepsilon(t) = \delta(t)$.

Notation: $u_\alpha := \lim_{\varepsilon\to 0+} P_\varepsilon(\phi_{\alpha,\varepsilon})$.

Lemma: (Scherzer & A.L., I.P. 03)

Let ϕ be a smooth function, having no critical point in a compact neighborhood M of the level set $\phi^{-1}(0)$. Then,

$$\lim_{\varepsilon \to 0+} P'_\varepsilon(\phi) = \delta(\phi)/|\nabla\phi| \ .$$

($\delta(\phi)$ is the one-dimensional δ-distribution centered at the level line in normal direction)

From the definition of a minimizer of (6) it follows that for all $h \in H^1(\Omega)$:

$$\int_\Omega (F(u_{\varepsilon,\alpha}) - y) F'(u_{\varepsilon,\alpha}) P'_\varepsilon(\phi_{\varepsilon,\alpha}) h \ + \\ + \alpha \int_\Omega ((\phi_{\varepsilon,\alpha} - \phi_*)h + \nabla(\phi_{\varepsilon,\alpha} - \phi_*)\nabla h) = 0. \qquad (7)$$

(note that $P'_\varepsilon(\phi)$ is self-adjoint)

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The optimality condition in (7) can be rewritten in the form:

$$
\begin{aligned}
P_\varepsilon'(\phi_{\varepsilon,\alpha})F'(u_{\varepsilon,\alpha})^*(F(u_{\varepsilon,\alpha})-y)+\alpha(I-\Delta)(\phi_{\varepsilon,\alpha}-\phi_*) &= 0 \text{ on } \Omega\,,\\
\frac{\partial(\phi_{\varepsilon,\alpha}-\phi_*)}{\partial n} &= 0 \text{ on } \partial\Omega\,.
\end{aligned}
\tag{8}
$$

Thus $u_\alpha = \lim_{\varepsilon\to 0+} u_{\varepsilon,\alpha}$ and $\phi_\alpha = \lim_{\varepsilon\to 0+} \phi_{\varepsilon,\alpha}$ satisfy

$$
\delta(\phi_\alpha)\frac{F'(u_\alpha)^*(F(u_\alpha)-y)}{|\nabla\phi_\alpha|}+\alpha(I-\Delta)(\phi_\alpha-\phi_*)=0\,. \tag{9}
$$

From (9) follows that

$$
(I-\Delta)^{-1}\left(\delta(\phi_\alpha)\frac{F'(u_\alpha)^*(F(u_\alpha)-y)}{|\nabla\phi_\alpha|}\right)+\alpha(\phi_\alpha-\phi_*)=0\,.
$$

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Set $\alpha = \frac{1}{\Delta t}$, $\phi_\alpha = \phi(t)$, $\phi_* = \phi(0)$, $u(t) = P(\phi(t))$.

By taking the formal limit $\Delta t \to 0+$ we get the asymptotic regularization method

$$\frac{\partial \phi}{\partial t} = -(I - \Delta)^{-1} \left(\delta(\phi(t)) \frac{F'(u(t))^*(F(u(t)) - y)}{|\nabla \phi(t)|} \right) . \tag{10}$$

The left hand side $v = \partial\phi/\partial t$ of (10) solves the equation

$$\begin{aligned} (I - \Delta)v &= -\delta(\phi(t)) \frac{F'(u(t))^*(F(u(t)) - y)}{|\nabla \phi(t)|} , \\ \partial v / \partial n &= 0 . \end{aligned} \tag{11}$$

(v is the *velocity* for the evolution of the Level-Set function)

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Relation to other Level-Set Methods:

(10) is a Hamilton-Jacobi type equation of the form

$$\partial\phi/\partial t + V\nabla\phi = 0 \tag{12}$$

with velocity $V = \dfrac{(I-\Delta)^{-1}\left(\delta(\phi)\frac{F'(u(t))^*(F(u(t))-y)}{|\nabla\phi(t)|}\right)}{|\nabla\phi(t)|}\dfrac{\nabla\phi}{|\nabla\phi(t)|}.$

The *classical* level-set method considered by Santosa'96 uses the velocity choice

$$V = -F'(u(t))^*(F(u(t))-y)\nabla\phi(t)/|\nabla\phi(t)|\ .$$

The differential equation $\dfrac{\partial\phi}{\partial t} = F'(u(t))^*(F(u(t))-y)|\nabla\phi(t)|$
is solved explicit in time, which results in

$$\frac{\phi(t+\Delta t)-\phi(t)}{\Delta t} = F'(u(t))^*(F(u(t))-y)|\nabla\phi(t)|.$$

Relation to Shape Optimization:

Lemma: (Scherzer & A.L., I.P. 03)

The term

$$\delta(\phi)\frac{F'(u)^*(F(u)-y)}{|\nabla\phi|}$$

is the steepest descent direction of the functional $\|F(u)-y\|^2$ with respect to the *shape* of the level set $\phi^{-1}(0)$.

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