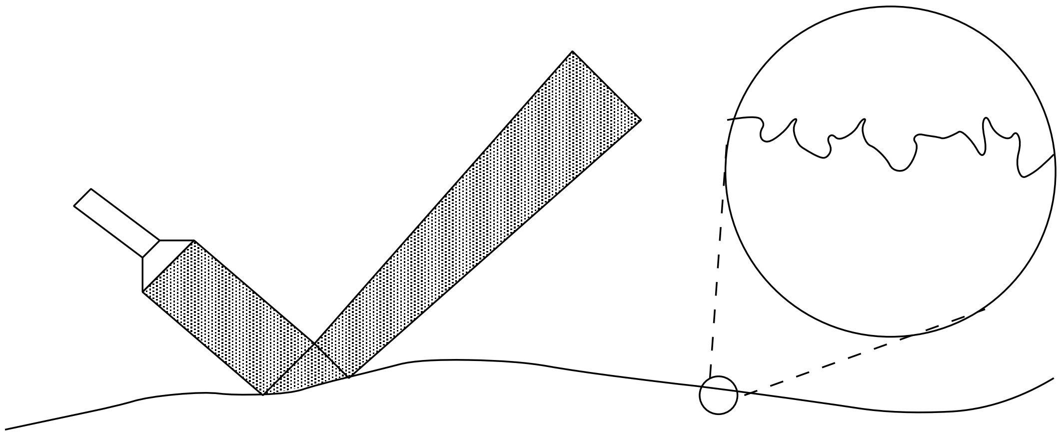
Inverse Schrödinger scattering problem for random potentials in plane

Matti Lassas, Lassi Päivärinta, Eero Saksman

1 Problem in random medium

Flashlight paradox:



2 Schrödinger equation

Let $D \subset \mathbf{R}^2$ be a compact domain, $\operatorname{supp}(q) \subset D$,

$$
\begin{aligned}
&(-\Delta + q(x) - k^2)u(x) = \delta_y(x) \quad \text{in } \mathbf{R}^2, \\
&u \text{ satisfies radiation condition.}
\end{aligned}
$$

When $u_{in} = G(x, y, k)$ is Green's function on $\mathbf{R}^2$, we write

$$
\begin{aligned}
u &= u_{in} + u_{sc}, \\
u_{sc} &= u_{sc}(x, y, k).
\end{aligned}
$$

Deterministic inverse problems

1. Rigorous results applying harmonic and microlocal analysis. Analyzed by Sylvester-Uhlmann, Novikov-Khenkin, Nachman, Brown-Uhlmann, Astala-Päivärinta etc.

2. Reconstruction algorithms very non-linear.

Inverse problems in random media

1. Scatterer can be very complicated. The inverse problem is to find characteristic properties like correlation length etc.

2. Results use often approximations

3. Reconstruction algorithms often linear.

4. Asymptotic methods by Papanicolaou et al, e.g. $\rho = \rho(x/\epsilon^2)$, $k = k_0/\epsilon$.

Our aim is to combine these approaches.

Problem: Consider the stochastic process $q(x,\omega)$. Let $q(x,\omega_0)$ be a realization of a random scatterer.
Do measurements from $q(x,\omega_0)$ determine the parameters of the stochastic process $q(x,\omega)$?

Motivation: 1-dimensional example:
$Q(t)$ satisfies stochastic differential equation

$$Q(t) = \int_0^t \alpha(s) dW(s), \quad \text{where } dW(t) \text{ is white noise.}$$

Covariance function of $Q(t)$ is

$$\mathbb{E}(Q(t_1)Q(t_2)) = \int_0^{\min(t_1,t_2)} \frac{\alpha(s)^2}{2}\, ds.$$

When t_2 is fixed,

$$\mathbb{E}(Q(t_1)Q(t_2)) = \alpha(t_2)^2 |t_1 - t_2| + C^{1,1}\text{-function.}$$

3 Definition of $q(x, \omega)$

We assume

1. $q(x, \omega)$ is a Gaussian generalized function in $D \subset \mathbf{R}^2$.
2. $q(x, \omega)$ is microlocally isotropic.

Definition 1. $q(x)$ is a Gaussian generalized function in $D \subset \mathbf{R}^2$ if for any $\phi \in C_0^\infty(D)$

$$\langle q, \phi \rangle \text{ is Gaussian.}$$

We assume that

$$\mathbb{E}(q(x)) = 0, \quad \mathbb{E}(q(x)q(y)) = C(x, y).$$

The covariance operator C of q is defined by

$$\langle C\phi, \psi \rangle = \int_{D \times D} C(x, y)\phi(x)\psi(y)\, dxdy.$$

Definition 2.
q is microlocally isotropic (of order s) if its covariance operator C is a pseudo-differential operator with symbol

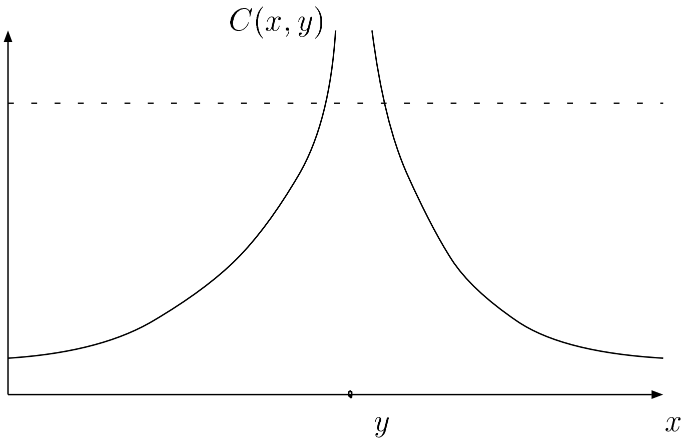
$$\sigma(C)(x,\xi) = \mu(x)|\xi|^{-s} + r(x,\xi), \quad \mu \in C_0^\infty(D),$$

where $r(x,\xi) \in S^{-s-1}(D \times \mathbf{R}^2)$.

Let $s = 2$. When y is fixed,

$$C(x,y) = -\mu(y) \log|x-y| + \mathcal{O}(1).$$

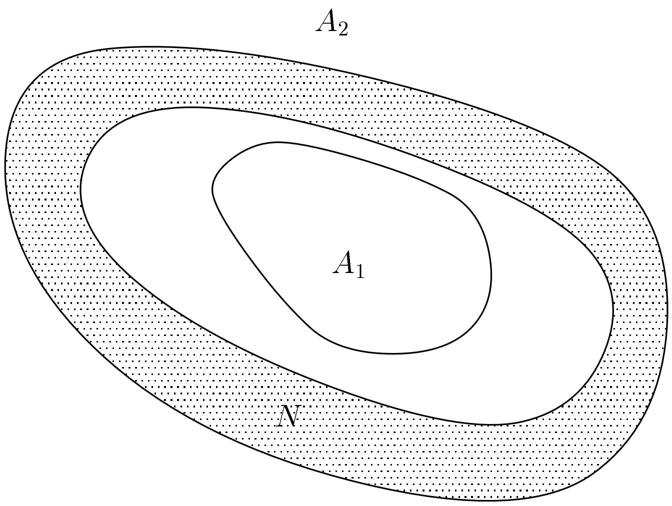
We say that $\mu(x)$ is the correlation strength function of random field q. It is related to correlation length of q.



4 Example

Definition 2. $q(x)$ is Markov random field if

$q|_{A_1}$ and $q|_{A_2}$ are independent on condition that $q|_N$ is given.



Markov property implies that $C(x, y)$ is a Green's function of an elliptic differential operator $P(x, D)$. If

$$P(x, y) = -\nabla \cdot a(x) \nabla + b(x),$$

then q is microlocally isotropic.

5 The main result

Theorem 1 *Let $U \subset \mathbf{R}^2 \setminus D$ be convex and q be microlocally isotropic field in D of order 2. Then*

1. *The measurement*

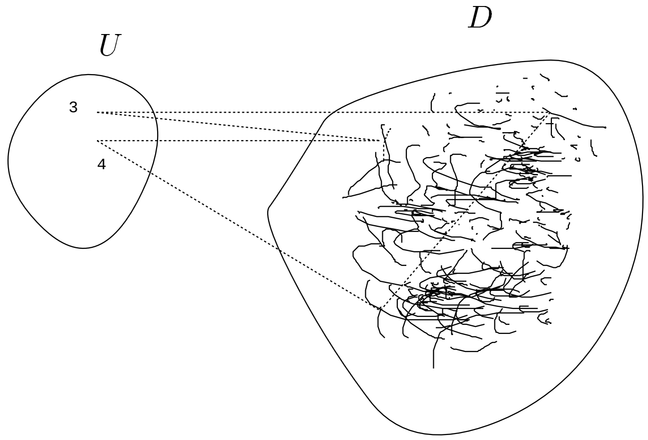
$$m(x, y, \omega) = \lim_{N \to \infty} \frac{1}{N-1} \int_1^N |u_{sc}(x, y, k, \omega)|^2 \, k^4 dk$$

exists almost surely.

2. *The measurement is a deterministic function,*

$$m(x, y, \omega) = m_0(x, y) \quad a.s.$$

3. *The backscattering data $m_0(x, x)$, $x \in U$ determines uniquely $\mu(x)$. The reconstruction is linear.*



Main steps of the proof: Let

$$G_k u(x) = \int_{\mathbf{R}^2} G(x,y,k)u(y)\,dy,$$

$$G(x,y,k) = \frac{i}{4} H_0^{(1)}(k|x-y|),$$

$$u_0 = u_{in} = G(x,y,k), \quad u_{n+1} = G_k(qu_n).$$

Then we have the Born series

$$u_{sc} = u_1 + u_2 + u_3 + \dots$$

1. We have $q \in W_0^{-\varepsilon,p}(D)$ for all $\varepsilon > 0$ and $p < \infty$ a.s.

Lemma 1 *Scattering problem with $q \in W_0^{-\varepsilon,p}(D)$, $p > n/2$, $0 < \epsilon < \frac{n}{4}(\frac{2}{n} - \frac{1}{p})$ is uniquely solvable.*

Proof: Consider Lippmann-Schwinger equation

$$(I + G_k q)u_{sc} = -G_k q u_{in}. \tag{1}$$

Using

1. Estimates for products of Sobolev functions,
2. Agmon-type estimate for $||G_k||$, $G_k : W_0^{-\varepsilon,p_1}(D) \to W^{\varepsilon,p_2}(D)$.
3. Unique continuation using complex geometrical optics

one shows that (1) is uniquely solvable and

$$||u_n|| \leq ck^{-\alpha n}, \quad \alpha > 0.$$

This implies that u_{sc} exists a.s. and that $u_3 + u_4 + \ldots$ are negligible in measurement.

Lemma 2 *For $k_1, k_2 > 1$,*

$$|\mathbb{E}(u_1(x,x,k_1)\overline{u_1(x,x,k_2)})| \le \frac{c_N}{k_1^2 k_2^2 (1+|k_1-k_2|^N)}.$$

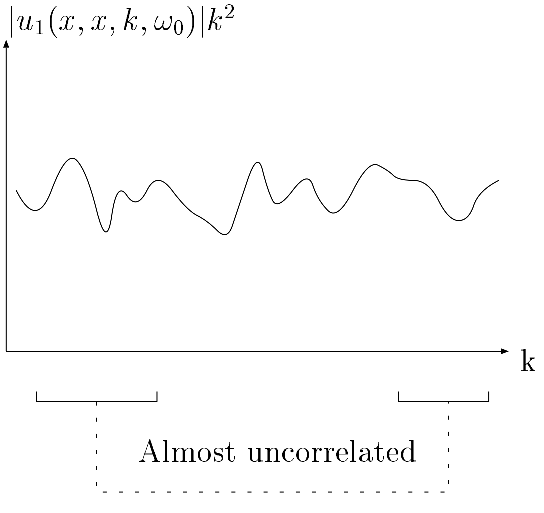
Proof: Covariance $C(z_1,z_2)$ is a conormal distribution and

$$\mathbb{E}(u_1(x,x,k_1)\overline{u_1(x,x,k_2)}) \approx$$

$$\int_{D\times D} e^{2i(k_1|x-z_1|-k_2|x-z_2|)} \frac{C(z_1,z_2)}{k_1k_2|x-z_1|\,|x-z_2|}\, dz_1 dz_2$$

$$= \frac{1}{k_1k_2}\int_{\mathbf{R}^2} e^{i(k_1-k_2)s_1} c(s_1,s_2;(k_1+k_2),0)\, ds_1 ds_2$$

where $c(s;\xi)$ is the symbol of $C(z_1,z_2)|z_1-x|^{-1}|z_2-x|^{-1}$ in appropriate coordinates (s_1,s_2).



3. Consider term u_2 in Born series.

Lemma 3 *For all $x \in U$ it holds almost surely that*

$$\lim_{K\to\infty} \frac{1}{K-1} \int_1^K |k^2 u_2(x,x,k,\omega)|^2 dk = 0.$$

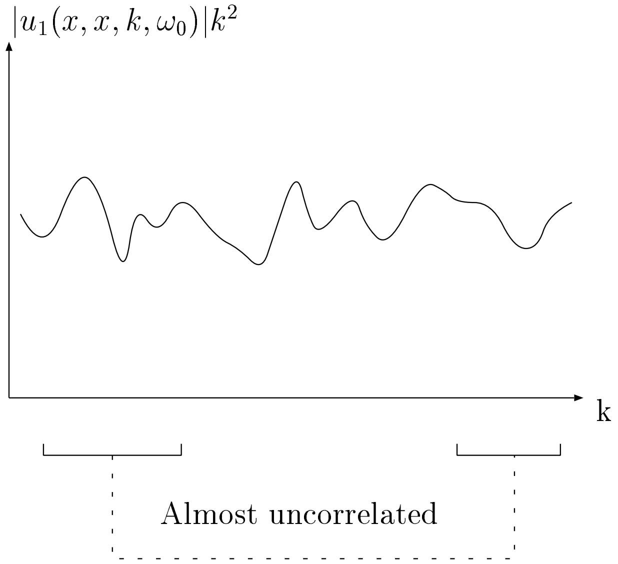
Proof: We obtain that

$$\mathbb{E} \int_1^\infty |k^{3/2} u_2(x,x,k)|^2 dk \leq C.$$

Thus

$$\frac{1}{K} \int_1^K |k^2 u_2(k)|^2 dk \;=\; \int_1^K \frac{k}{K} |k^{3/2} u_2(k)|^2 dk \to 0$$

by dominated convergence theorem.



This implies that the measured signal is ergodic:

Lemma 4 *For almost every $\omega \in \Omega$*

$$\lim_{N\to\infty} \frac{1}{N-1} \int_1^N |u_1(x,x,k,\omega)|^2 k^4 dk$$
$$= \lim_{k\to\infty} \mathbb{E}(|u_1(x,x,k)|^2 k^4).$$

Moreover,

$$m_0(x,x) = \lim_{k\to\infty} \mathbb{E}(|u_1(x,x,k)|^2 k^4) = \int_D \frac{\mu(z)}{2|x-z|^2} dz.$$

4. Use of integral geometry.

We can see that the backscattering data determines integrals over circles. Then the reconstruction procedure is

$$m_0(x,x) \longrightarrow \int_{|z-x|=R} \mu(z)dS_z \longrightarrow \mu(x).$$