Electrostatic Imaging via Conformal Mapping

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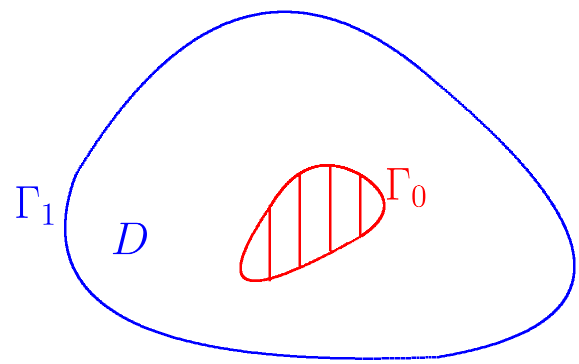
Or: **A new solution method for inverse boundary value problems for the Laplace equation**

Determine shape Γ_0 of a

- **perfectly conducting** or
- **nonconducting** inclusion or
- inclusion with **different conductivity**

from overdetermined Cauchy data on Γ_1



Applications in the field of **nondestructive testing** via **electrostatic imaging** or **thermal imaging**, e.g., **impedance tomography**

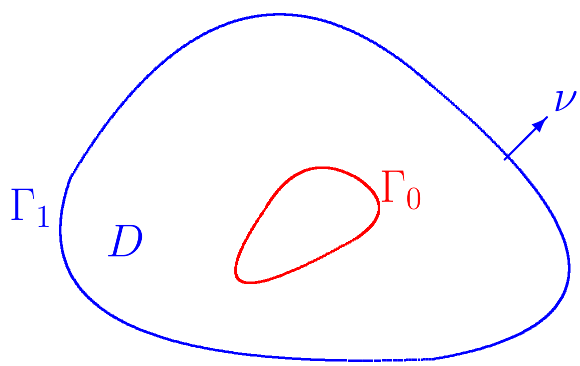
Here: Perfectly conducting inclusion, i.e.,
inverse Dirichlet problem
Extensions to other boundary conditions are in preparation

1. Brief survey on other methods
(see 2. ed. of Linear Integral Equations)

2. Description of new method

3. Some numerical examples

The inverse problem



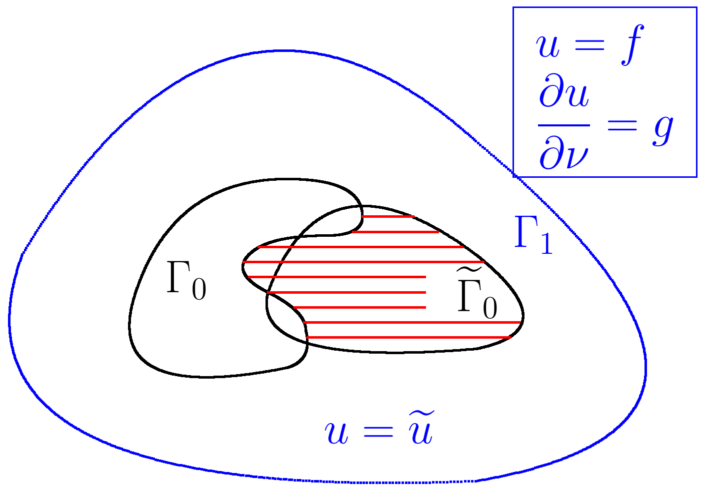
$$
\begin{aligned}
\Delta u &= 0 \quad \text{in } D \\
u &= 0 \quad \text{on } \Gamma_0 \\
u &= f \quad \text{on } \Gamma_1
\end{aligned}
$$

Inverse Problem:

Given $g = \dfrac{\partial u}{\partial \nu}$ on Γ_1 (and f), **find** boundary Γ_0

Uniqueness!!!

Uniqueness!!!



$\Gamma_0 \leftrightarrow u$

$\widetilde{\Gamma}_0 \leftrightarrow \widetilde{u}$

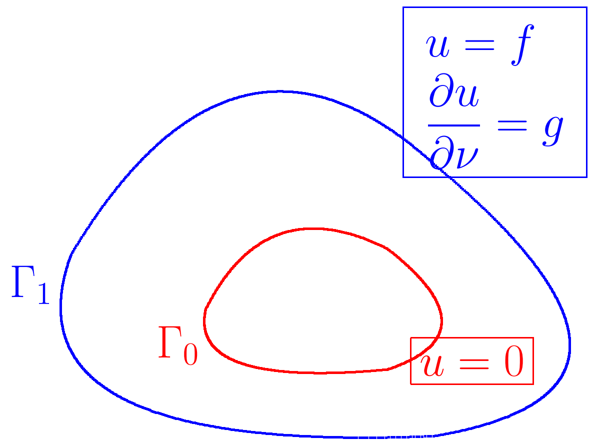
In shaded domain: $\triangle u = 0$

On boundary: $u = 0$

Schiffer ≈ 1960

Existence???

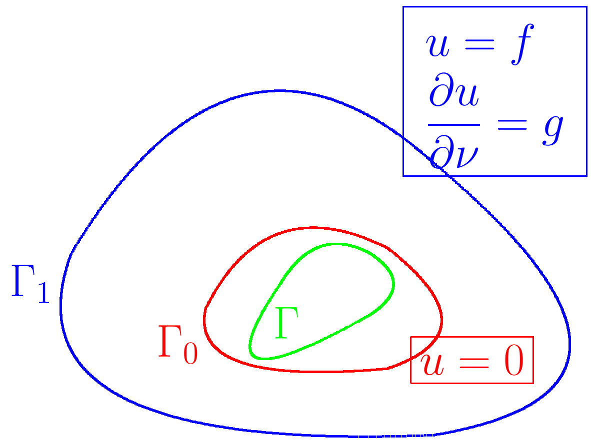
For inverse boundary value problems, in general, **wrong question** to ask. Would need to characterize Cauchy data on Γ_1 for which the corresponding solution vanishes on a closed surface Γ_0 (or curve) within Γ_1.



Main Task: Assuming **correct data** or **perturbed** correct data,
design method for approximate and stable solution

Separate ill-posedness and nonlinearity

1. Determine u from Cauchy data on Γ_1 (for example via potentials and integral equations of the first kind)
Kirsch, K. (1987)



2. Find Γ_0 as location of the zeros of u (in a least squares sense)

Pros:

- Conceptionally simple
- No need for forward solver

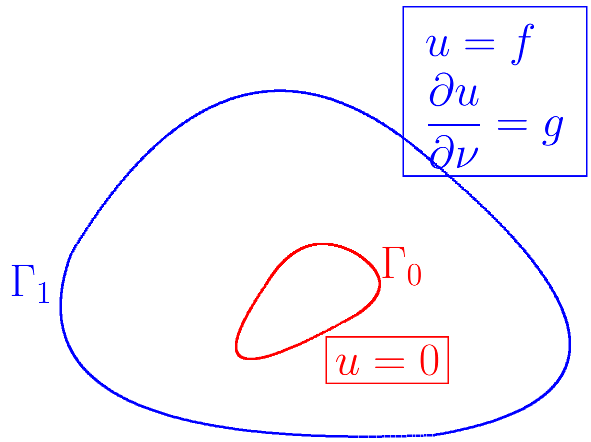
Contras:

- No high class reconstructions
- Gap between theory and numerics
- Domain for Cauchy problem not known in first step

Newton type iterations

1. Interpret inverse problem as operator equation $F(\Gamma_0) = g$ where

$$F : \Gamma_0 \mapsto \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_1}$$



2. Solve by regularized Newton iterations

Pros:
- Conceptionally simple
- High class reconstructions

Contras:
- Need forward solver
- Need good a priori information
- Convergence analysis difficult

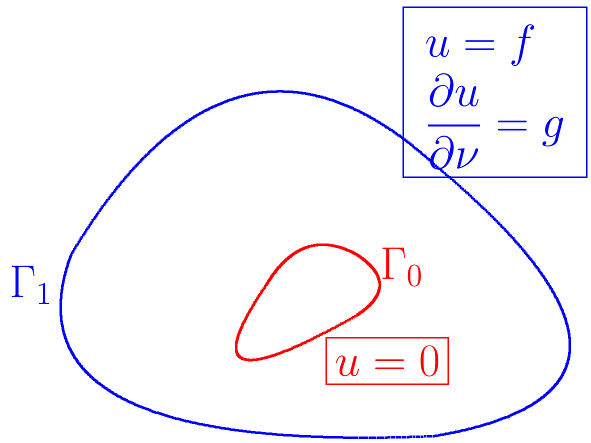
Hohage (1999), **Potthast** (2001)

Kirsch's factorization method

Characterize unknown domain via spectral data of the Dirichlet-to-Neumann operator

$$A : u \mapsto \frac{\partial u}{\partial \nu}$$

on Γ_1



Hähner (1999), **K.** (1999), **Kühn** (2001)
Brühl, Hanke (2000)

Pros:

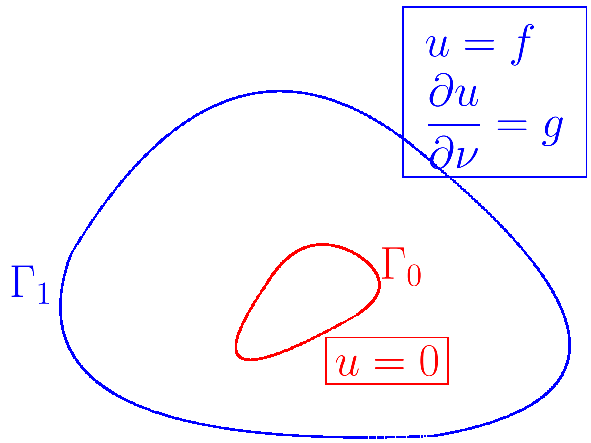
- Elegant mathematics
- Simple implementation
- No a priori information needed

Contras:

- Need a lot of data
- No high class reconstructions
- Very sensitive to noise

Our method

1. Solve nonlocal nonlinear ordinary differential equation for boundary values of a conformal mapping (by successive iterations)



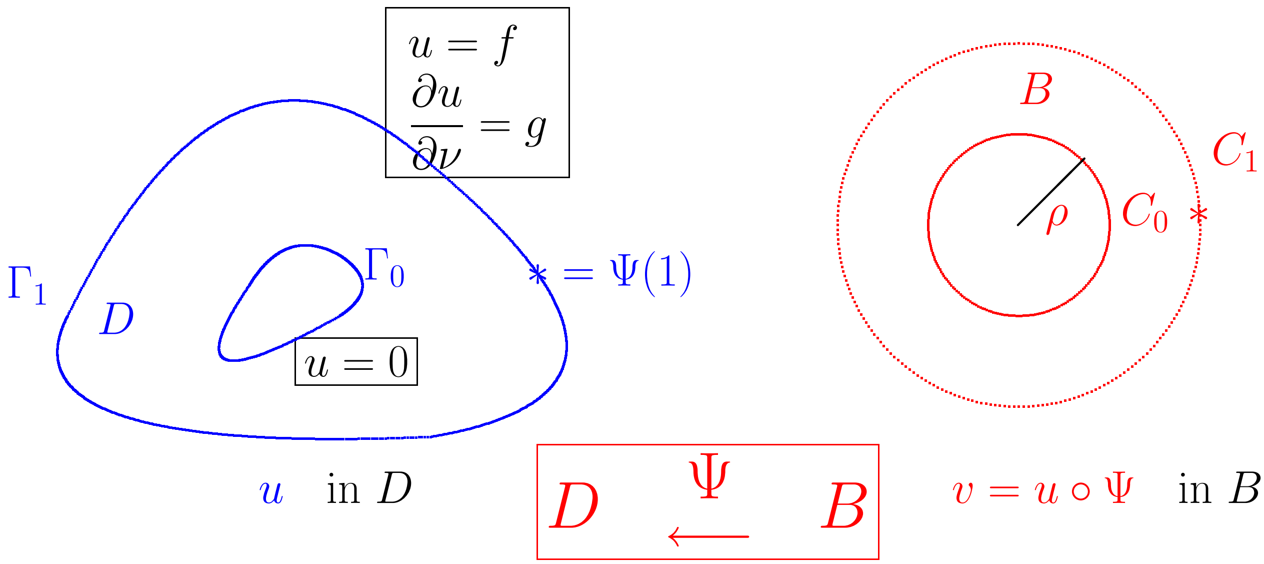
2. Solve Cauchy problem for a holomorphic function in an annulus

Pros:

- Conceptionally simple
- Satisfactory reconstructions
- Domain for Cauchy problem is known

Contras:

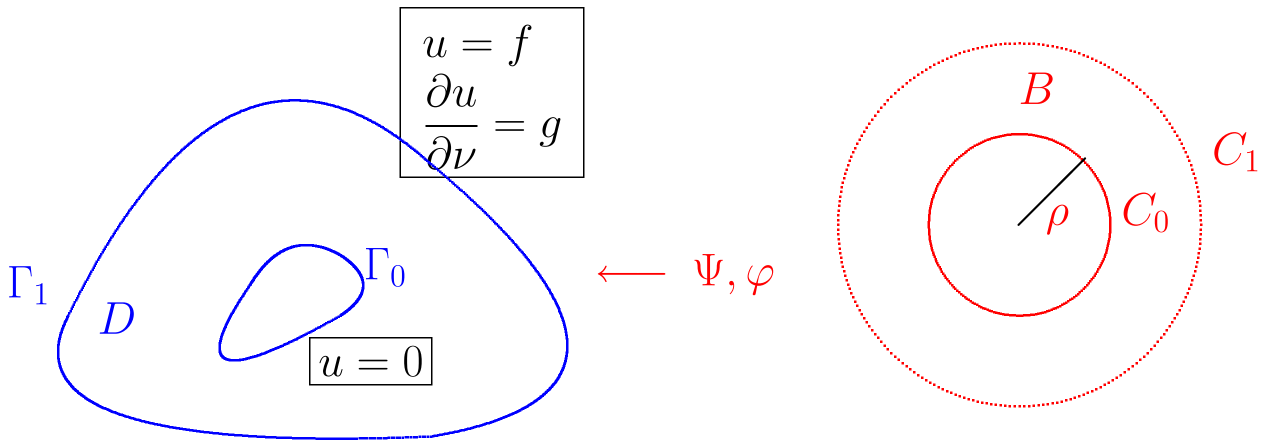
- Restricted to two dimensions
- and to Laplace equation



$\Gamma_1 = \{\gamma(s) : s \in [0, L]\}$ $\quad$ $C_1 = \{e^{it} : t \in [0, 2\pi]\}$

φ : arc length on C_1 $\quad \mapsto \quad$ arc length on Γ_1

$\boxed{\Psi(e^{it}) = \gamma(\varphi(t))}$ $\quad$ knowing φ equivalent to knowing $\Psi|_{C_1}$



$u, \tilde{u}$ and $v = u \circ \Psi, \tilde{v} = \tilde{u} \circ \Psi$ conjugate harmonics

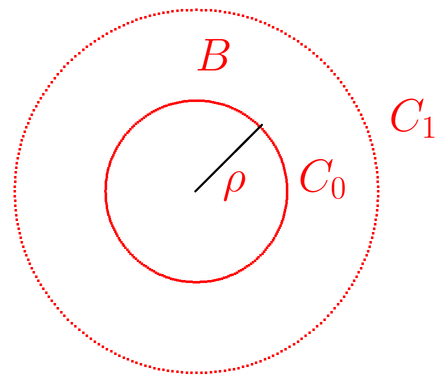
$$\tilde{v}(t) = \tilde{u}(\varphi(t)) \quad \Rightarrow \quad \frac{\partial \tilde{v}}{\partial t} = \frac{\partial \tilde{u}}{\partial s} \frac{d\varphi}{dt}$$

Cauchy–Riemann equations $\quad \Rightarrow \quad \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} \frac{d\varphi}{dt}$

$$\boxed{\frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi}}$$

$A =$ Dirichlet-to-Neumann map for B

$$u = f, \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_1$$



$$\int_{C_0} \frac{\partial v}{\partial \nu}\, ds = \int_{C_1} \frac{\partial v}{\partial \nu}\, ds = \int_{\Gamma_1} g\, ds$$

$$\int_{C_0} \left\{ \ln|x|\, \frac{\partial v}{\partial \nu} - v\, \frac{1}{|x|} \right\} ds = \int_{C_1} \left\{ \ln|x|\, \frac{\partial v}{\partial \nu} - v\, \frac{1}{|x|} \right\} ds$$

$$\ln \rho \int_{C_0} \frac{\partial v}{\partial \nu}\, ds = -\int_{C_1} v\, ds$$

$$\boxed{\rho = \exp\left(-\frac{\int_0^{2\pi} f \circ \varphi\, dt}{\int_{\Gamma_1} g\, ds} \right)}$$

$$\frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi}$$

A = Dirichlet-to-Neumann map for B

$$L = \int_0^{2\pi} \frac{A(f \circ \varphi)}{g \circ \varphi}\, dt$$

$$\frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi} + \frac{L}{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \frac{A(f \circ \varphi)}{g \circ \varphi}\, dt$$

Makes sure that

$$\varphi(2\pi) = L$$

throughout the iteration

$$\rho = \exp\left(-\frac{\int_0^{2\pi} f \circ \varphi \, dt}{\int_{\Gamma_1} g \, ds}\right)$$

$$\frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi} + \frac{L}{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \frac{A(f \circ \varphi)}{g \circ \varphi} \, dt$$

$$\rho_n = \exp\left(-\frac{\int_0^{2\pi} f \circ \varphi_n \, dt}{\int_{\Gamma_1} g \, ds}\right)$$

$$\frac{d\varphi_{n+1}}{dt} = \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} + \frac{L}{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} \, dt$$

$$\varphi_0(t) = \frac{L}{2\pi}\, t, \quad \text{correct if } D \text{ annulus}$$

Theorem Under appropriate assumptions on D and f the successive approximations converge in $H^1[0, 2\pi]$.

$$\frac{d\varphi_{n+1}}{dt} = \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} + \frac{L}{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n}\, dt$$

Numerical implementation:

$$\varphi_n(t) \approx \frac{L}{2\pi}\, t + \alpha_{n,0} + \sum_{k=1}^{N} [\alpha_{n,k} \cos kt + \beta_{n,k} \sin kt]$$

$$L_n := \frac{1}{2\pi}\int_0^{2\pi} \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n}\, dt - \frac{L}{2\pi}$$

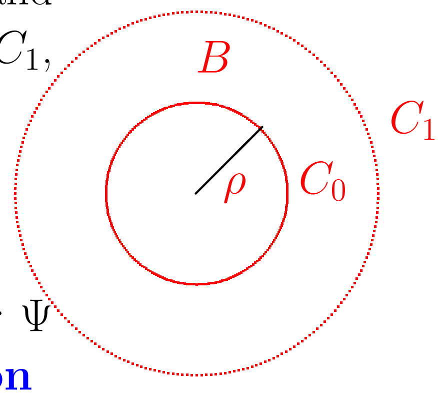
$$(g \circ \varphi_n)(t_j)\, \{\varphi'_{n+1}(t_j) + L_n\} - A_n(f \circ \varphi_n)(t_j) = 0, \qquad j = 1, \ldots, J$$

Solve by least squares to update coefficients

Use trigonometric interpolation for $f \circ \varphi$ and $g \circ \varphi$

Now, we know the radius ρ and $\Psi = \gamma \circ \varphi$ on the outer circle C_1, i.e., Fourier series

$$\gamma(\varphi(t)) = \sum_{k=-\infty}^{\infty} b_k e^{ikt}$$



Solve **Cauchy problem** for Ψ in B via **Laurent expansion**

$$\Psi(z) = \sum_{k=-\infty}^{\infty} b_k z^k, \quad \rho \le |z| \le 1$$

and obtain unknown boundary by

$$\Gamma_0 = \Psi(C_0) \approx \left\{ \sum_{k=-M}^{M} \rho^k b_k e^{ikt}, \quad 0 \le t \le 2\pi \right\}$$

Need to truncate because of **exponential ill-posedness**.

Our method

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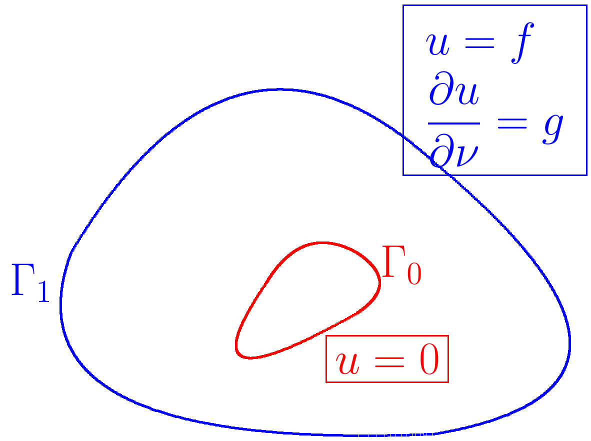


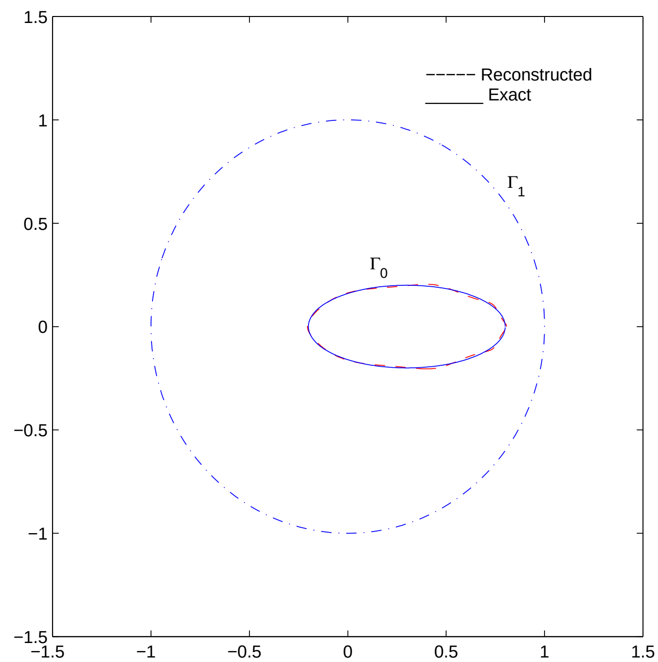
2. Solve Cauchy problem for a holomorphic function in an annulus

For numerical examples Γ_1 unit circle

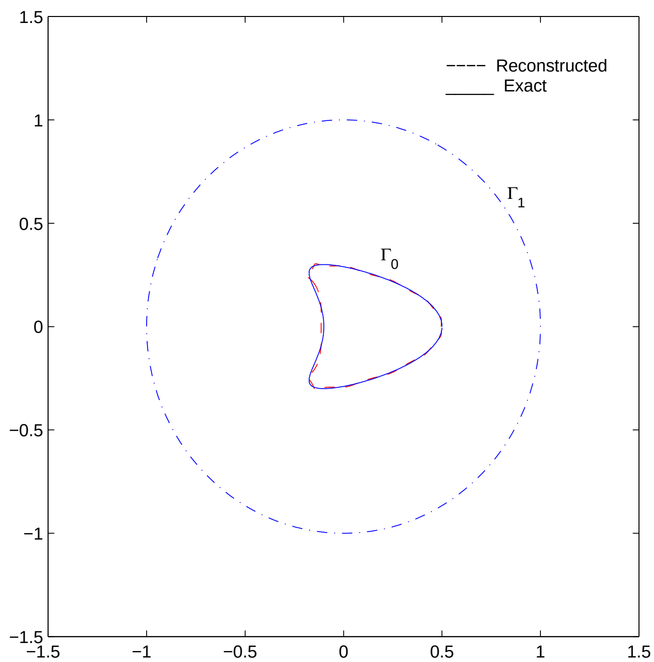
$$f(t) = 3 + 2\cos^2 t$$

$$N = 6 \ldots 8, \quad M = 6 \ldots 8, \quad n = 32$$

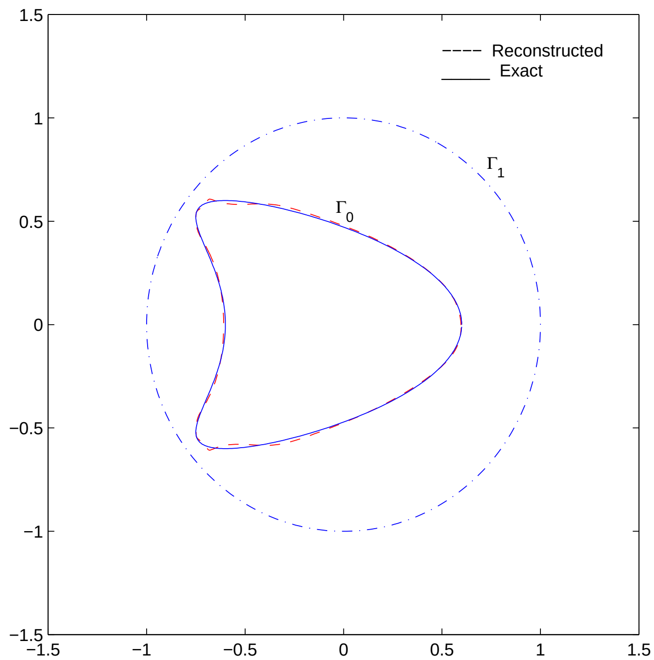
Between 6 to 10 iterations

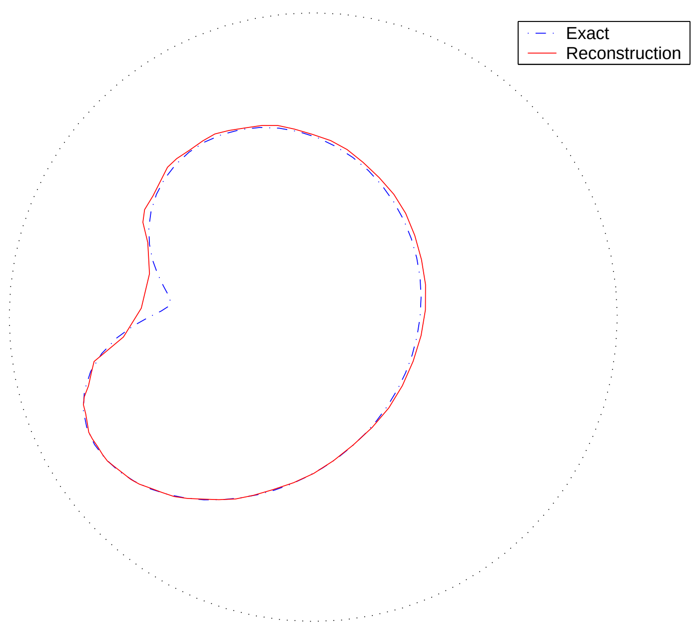


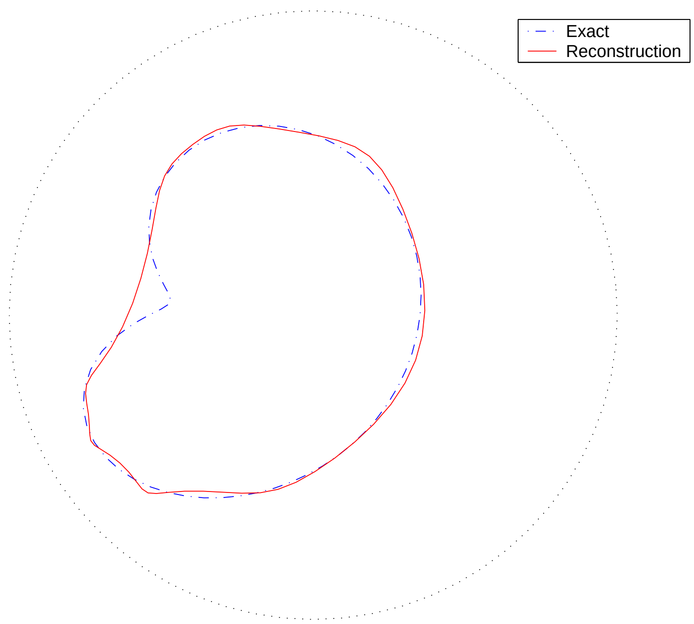
Reconstructed
Exact
Γ_1
Γ_0
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5



Reconstructed
Exact
Γ_1
Γ_0
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5


Reconstructed
Exact
Γ1
Γ0
1.5
1
0.5
0
−0.5
−1
−1.5
−1.5
−1
−0.5
0
0.5
1
1.5


Exact
Reconstruction


Exact
Reconstruction

Remarks:

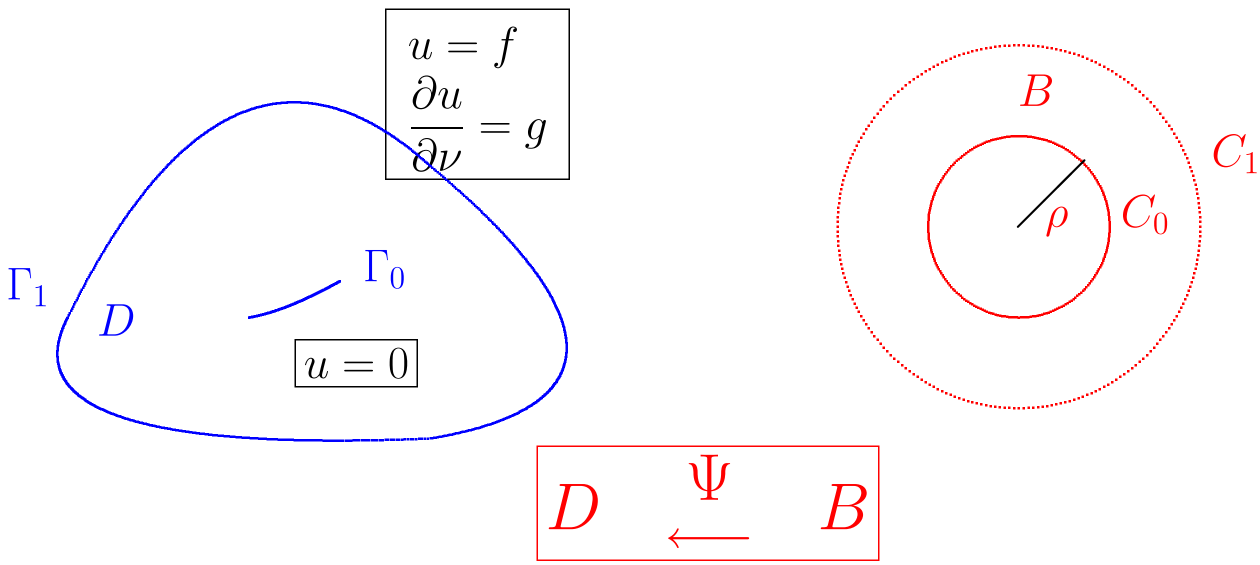
- **Idemen, Akduman** (1988)

$$f = f_0 = \text{const} \quad \Rightarrow \quad \boxed{\frac{d\varphi}{dt} = -\frac{1}{\ln\rho}\,\frac{f_0}{g\circ\varphi}}$$

- Nonconstant f required for extensions to other boundary conditions. However, no flux $\boxed{\int_{\Gamma_0} g\,ds = 0}$ has to be observed

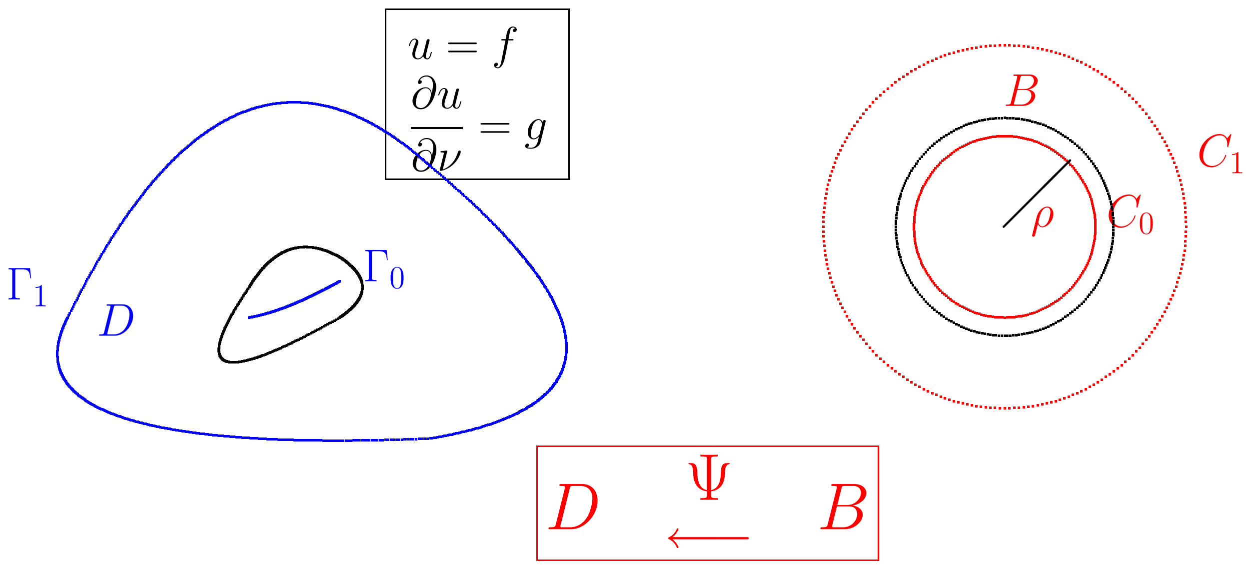
- **Inverse Problems 18, 1659–1672 (2002)**

Cracks!!!

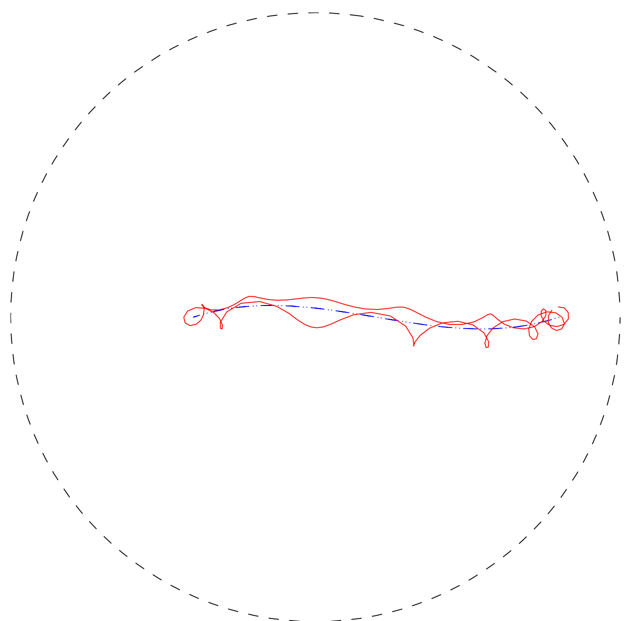


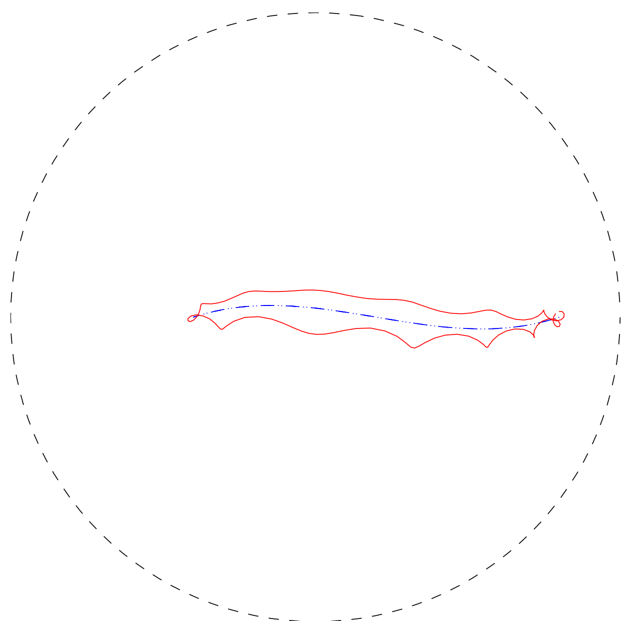
$u = f$
$\frac{\partial u}{\partial \nu} = g$
Γ_1
D
Γ_0
$u = 0$
B
C_1
ρ
C_0
$D \overset{\Psi}{\longleftarrow} B$

Cracks!!!



Try $\Gamma_0 \approx \Psi(\widetilde{C}_0)$ with radius $\rho(1+\lambda)$ for $\widetilde{C}_0$





Open problems:

- Other boundary conditions
- Incomplete data
- Satisfactory analysis for cracks
- Other regularizations in second step