Shape reconstruction from generalized brightness functions: extension of geometric tomography

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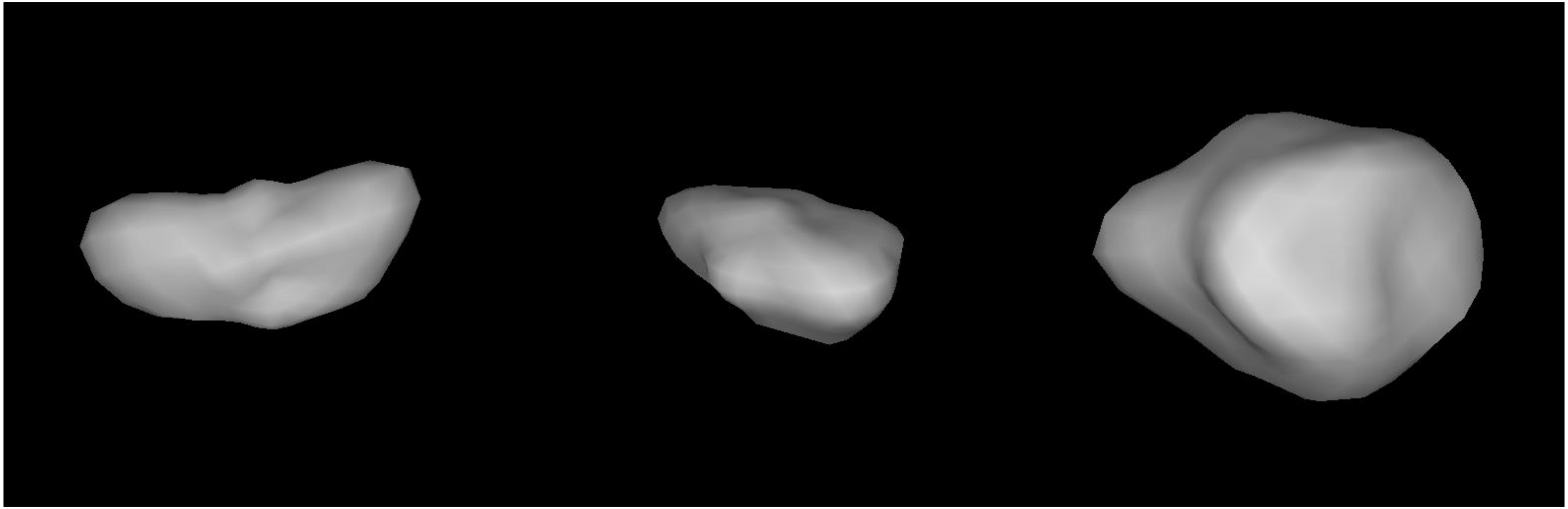


Joint work with, among others, Josef Durech (U. Prague), Daniel Hestroffer (U. Paris), Lars Lamberg (U. Helsinki), Chris Magri (U. Maine), Tadeusz Michalowski (A. Mickiewicz U.), Petr Pravec (U. Prague), Johanna Torppa (U. Helsinki), etc.

- "Geometric tomography" (Gardner et al.): how to reconstruct a body from (finitely many values of) the areas of its projections (brightness function)/cross-sections
- Nonunique for general objects (Blaschke's bullet/Reuleaux triangle); we only get the symmetric part of a convex object
- GENERALIZED brightness functions:
  1. Shadowing effects, i.e., the object is illuminated and viewed from different directions ($S^2 \times S^2$ instead of S^2)
  2. Light-scattering effects, i.e., additional weight functions in the projection integral
- Applications in solar system remote sensing: **The resolving capacity of photometric inversion lies between space telescope and radar, and its range extends from near-Earth to main-belt asteroids**

Brightness conribution dL of a visible and illuminated surface patch dA:

$$dL = S(\mu, \mu_0)\varpi \, dA \tag{1}$$

where S and ϖ are the scattering law (in a simple form here) and the intrinsic brightness of the surface material;

$$\mu = \mathbf{E} \cdot \mathbf{n}, \qquad \mu_0 = \mathbf{E}_0 \cdot \mathbf{n}, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{E}_0$ are, respectively, unit vectors towards the observer (Earth) and the Sun, and $\mathbf{n}$ is the surface unit normal. Often $\varpi \equiv 1$.

Geometric tomography is restricted to $\mathbf{E} = \mathbf{E}_0$ and $S = \mu$. In the general case, a typical form for S is, e.g.,

$$S = \mu\mu_0(\frac{1}{\mu + \mu_0} + c). \tag{3}$$

For convex bodies $\mathbf{r} = \mathbf{r}(\mathbf{n})$. When $\mathbf{n}$ is given by the polar coordinates $\mathbf{n} = \mathbf{n}(\vartheta, \psi)$, we have for the restricted S^2-case

$$L(\mathbf{E}, \mathbf{r}) = \int\int_{\mu \geq 0} S(\mu) \, G(\vartheta, \psi) \, \sin\vartheta \, d\vartheta \, d\psi, \tag{4}$$

where

$$G(\vartheta, \psi) = \frac{|\mathbf{J}(\vartheta, \psi)|}{\sin\vartheta}, \tag{5}$$

and $\mathbf{J}$ is the Jacobian

$$\mathbf{J}(\vartheta,\psi) = \frac{\partial \mathbf{r}}{\partial \vartheta} \times \frac{\partial \mathbf{r}}{\partial \psi}. \tag{6}$$

Expanding (L^2-function) G as a Laplace series

$$G(\vartheta,\psi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm}\, Y_l^m(\vartheta,\psi), \tag{7}$$

we obtain, with the projection direction given by (θ,φ),

$$L(\theta,\varphi) = 2\pi \sum_{lm} k_l\, g_{lm}\, Y_l^m(\theta,\varphi), \tag{8}$$

where, with $S = \mu^n$,

$$k_l^{(n)} = \int_0^1 x^n P_l(x)\, dx. \tag{9}$$

For integer n, $k_l^{(n)}$ have the recursion relation

$$k_l^{(n)} = \frac{n}{l+n+1}\, k_{l-1}^{(n-1)}, \tag{10}$$

starting with $k_0^{(n)} = \frac{1}{n+1}$, and for odd l $k_l^{(0)} = (-1)^{(l-1)/2} \frac{(l-2)!!}{(l+1)!!}$, while $k_l^{(0)} = 0$ for even $l > 0$.

Thus $k_l = 0$ for odd $l > 1$ for geometric tomography ($n = 1$), so the projections contain information only on g_{lm} with even l. If $n = 2$, $k_l = 0$ for even $l > 2$. Thus the scattering law must be *at least of the general form* (3) to hold complete information on convex shape on S^2-data.

1. General convex case

Y_{lm} are the basis functions of the irreducible representations of SO(3). We can also use the representations, i.e., the rotation matrices $D^{(l)}_{m'm}$, as our basis for describing $S^2 \times S^2$-data. Now the generalized brightness function is

$$L(\mathbf{E}, \mathbf{E}_0, \mathbf{r}) = \sum_{lm} g_{lm} \sum_{m'} D^{(l)}_{m'm}(\kappa, \epsilon, \delta)\, I_{lm'}(\alpha), \tag{11}$$

where the suitable Euler angles κ, ϵ, δ and the solar phase angle $\alpha = \arccos(\mathbf{E} \cdot \mathbf{E}_0)$ define a point on $S^2 \times S^2$, and

$$I_{lm'}(\alpha) = \int_\alpha^\pi \int_0^\pi S(\mu, \mu_0, \alpha)\, Y_l^{m'}(\vartheta, \psi)\, \sin\psi\, d\vartheta\, d\psi. \tag{12}$$

Due to the orthogonality of $D^{(l)}_{m'm}$, we know that observations at a given α can indeed be written as a series in $D^{(l)}_{m'm}(\kappa, \epsilon, \delta)$. Thus there are always equations directly relating a g_{lm} to an observed D-series coefficient $c^l_{m'm}$. There is always an $I_{lm'} \neq 0$ for any l, so information on all g_{lm} is preserved in L when $\alpha \neq 0$.

$=>$ $S^2 \times S^2$**-data uniquely determine the curvature of a convex surface.**

Minkowski has shown that the curvature of a convex surface uniquely determines its shape (Minkowski 1903, Nirenberg 1953).

$=>$ $S^2 \times S^2$**-data uniquely determine the shape of a convex surface.**

Minkowski problem

Define support function $\varrho(\vartheta, \psi) = \mathbf{r}(\vartheta, \psi) \cdot \mathbf{n}(\vartheta, \psi)$ (the inverse function for $\mathbf{r}$ is

$$\mathbf{r}(\vartheta, \psi) = M^T(\vartheta, \psi)\mathbf{q}(\vartheta, \psi), \tag{13}$$

where $\mathbf{q}^T = (\varrho_\vartheta, \varrho_\psi / \sin\psi, \varrho)$, and M is an orthogonal matrix), and the mixed volume of two bodies R and S

$$\tilde{V}(R,S) = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} \varrho_R(\vartheta,\psi)\, G_S(\vartheta,\psi)\, \sin\vartheta\, d\vartheta\, d\psi. \tag{14}$$

Let $V_R = 1$; then $\tilde{V}(R,S)$ reaches its minimum exactly when R is homothetic with S.

$\rightarrow$ minimize $< \mathbf{x}, \mathbf{g} >$ in R^n, with $V(\mathbf{x})$=1

$<=>$ maximize $V(\mathbf{x})$ while staying on the hyperplane $< \mathbf{x}, \mathbf{g} >= const.$ (now the constraint eqn. is linear). $\mathbf{x}$ and $\mathbf{g}$ are discretized either with Laplace series coeffs. (Lamberg and Kaasalainen 2001; Newton/steepest descent) or in polytope form with facet distances from the origin $\mathbf{l}$ and facet areas $\mathbf{A}$ for chosen facet normal set $\{\mathbf{n}_j, j = 1...n\}$ (using the R^3-dual space $\mathbf{r}'_j = \mathbf{n}_j / l_j$ in polytope construction from $\mathbf{l}$).

Using a Laplace series representation for ϱ, we have

$$\tilde{V}(R,S) = \sum_{lm} \varrho_{lm}^{(R)}\, a_{lm}^{(S)}, \tag{15}$$

where

$$a_{lm}^{(S)} = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} G_S(\vartheta,\psi)\, Y_l^m(\vartheta,\psi)\, \sin\vartheta\, d\vartheta\, d\psi. \tag{16}$$

In polytope representation, we have

$$\tilde{V}(R,S) = \frac{1}{3}\sum_{j=1}^{n} l_j^{(R)}\, A_j^{(S)}. \tag{17}$$

Brunn-Minkowski theorem: the set $\{l_j^{(R)} | \tilde{V}(R,R) \geq 1\}$ in R^n is convex, so a local minimum of $\tilde{V}(R,S)$ is the global one.

$\rightarrow$ demonstrably converging iteration procedure (Lamberg 1993).

Removing ill-posedness via (nonlinear) positive definiteness

The convex inverse problem can be cast in the form

$$\mathbf{L} = M\mathbf{g}. \tag{18}$$

Linear least-squares solution is ill-posed since it allows negative curvature or facet areas (but note that the integrals k_l and $I_{lm'}$ decrease fast with increasing $l \rightarrow$ high-order shape details affect L very little).

Stable result is obtained by making the curvature (or areas) positive definite by nonlinear minimization of

$$\chi^2 = ||\mathbf{L} - M\mathbf{g}||^2 \tag{19}$$

with $g_j = A_j = \exp(a_j)$ (conjugate gradients), or using

$$G(\vartheta, \psi) = \exp\Big(\sum_{lm} a_{lm} Y_l^m(\vartheta, \psi)\Big) \tag{20}$$

and Levenberg-Marquardt. Defining positivity is considerably more robust than, e.g., barrier functions – no additional reg. fns. are needed.

Similarly, any other parameters to be assigned to an interval $]a, b[$ are mapped by

$$a < x < b => x = a + (b-a)\frac{exp(y)}{1+exp(y)}, \qquad -\infty < y < \infty \tag{21}$$

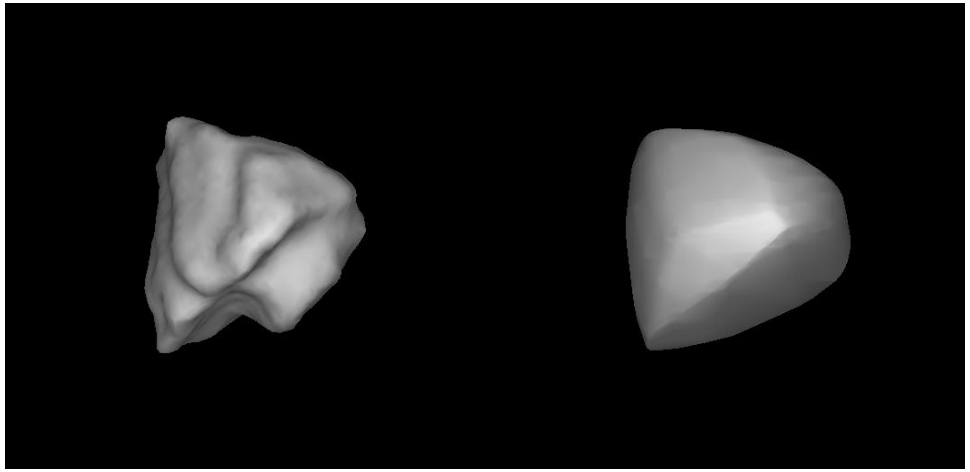
Indication of variegation of the intrinsic darkness of surface:

$$\sum_j \mathbf{n}_j A_j \neq \mathbf{0}. \tag{22}$$

This can also be used as a regularization function.

2. Nonconvex original → convex model

- This is where we have to leave analytic proofs because of nonconvexity
- Usually convex inversion produces a good approximation of the convex hull of the originating body
- The information content of brightness functions is strong: also the spin state of the body can be obtained (the rotation parameters that define the vectors $\mathbf{E}$ and $\mathbf{E}_0$ in the body's coordinate system)
- Minkowski stability: solution is stable against noise and insufficient knowledge of the scattering law



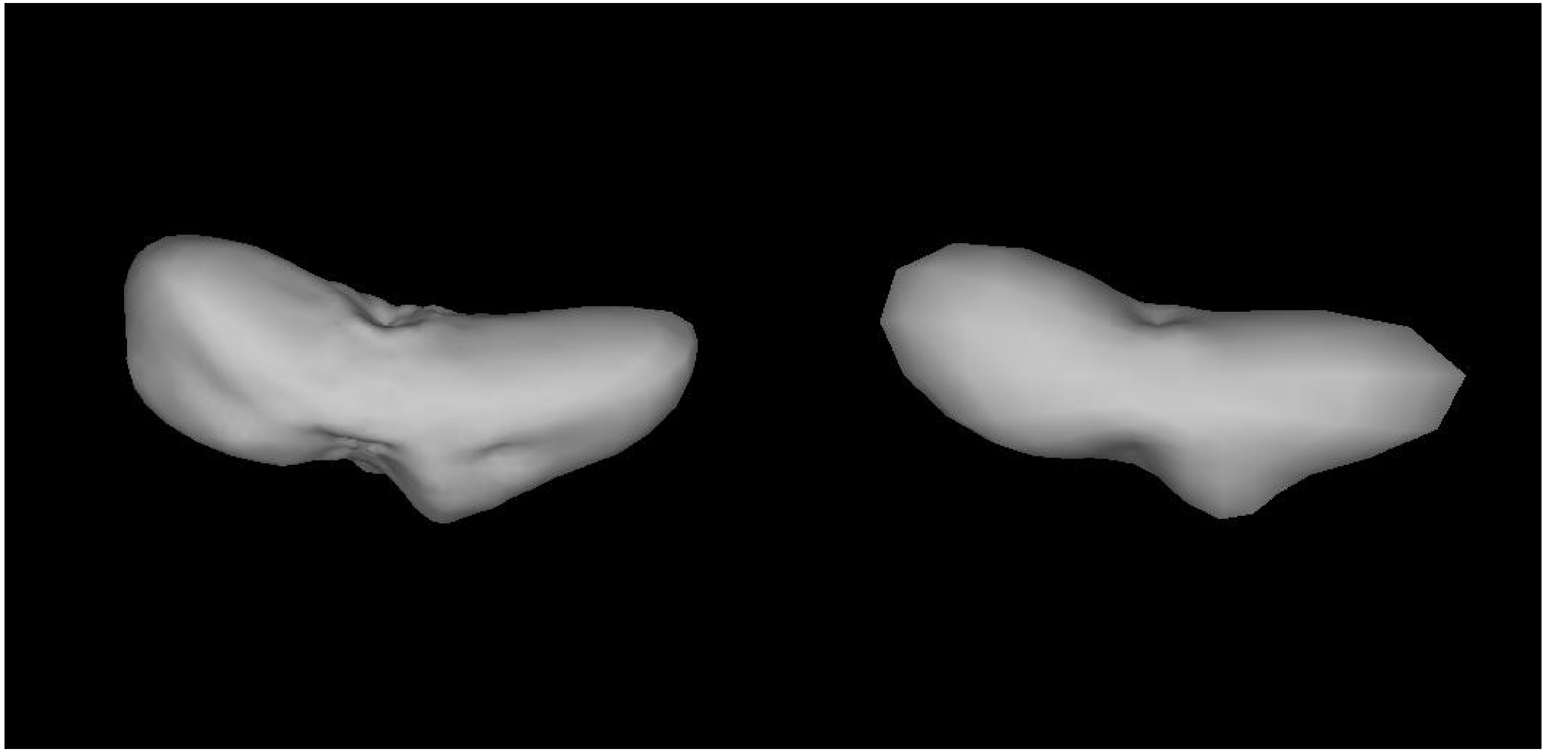
- Statistical properties of fractal/Gaussian/other surfaces at smaller scales than $\mathbf{r}$-resolution?

3. Nonconvex → nonconvex

- Optimize **r** directly with ray-tracing algorithms

Conjecture:

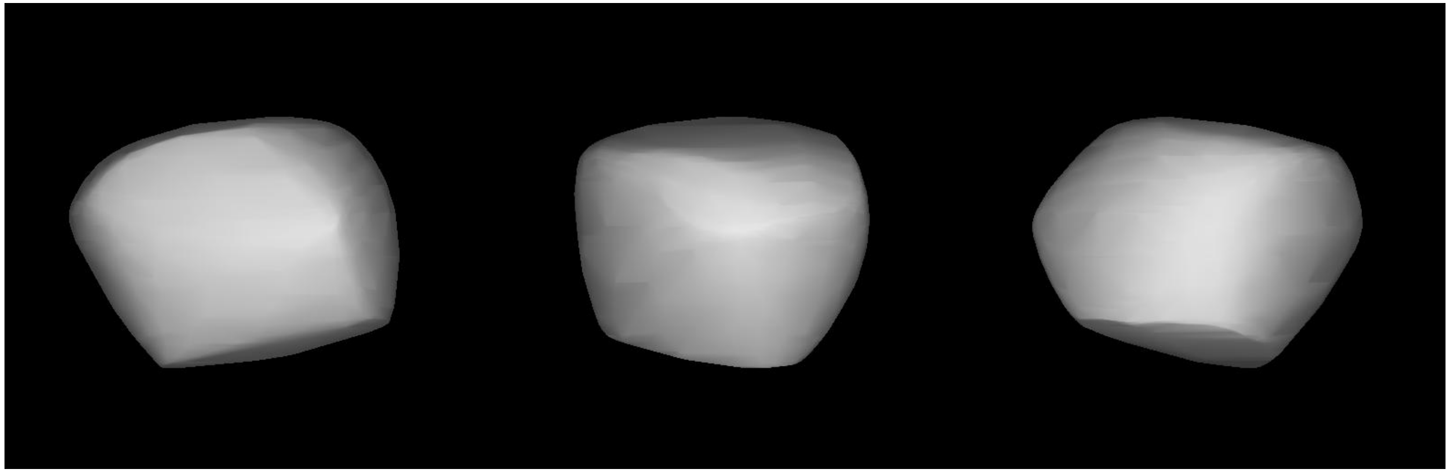
Generalized brightness functions determine the shape (and spin state) of a nonconvex body, with some possible constraints such as the difference between the viewing and illumination directions, the length scale for the reconstructed nonconvex details, and the coordinate system for the shape representation (but a class larger than star bodies is allowed).



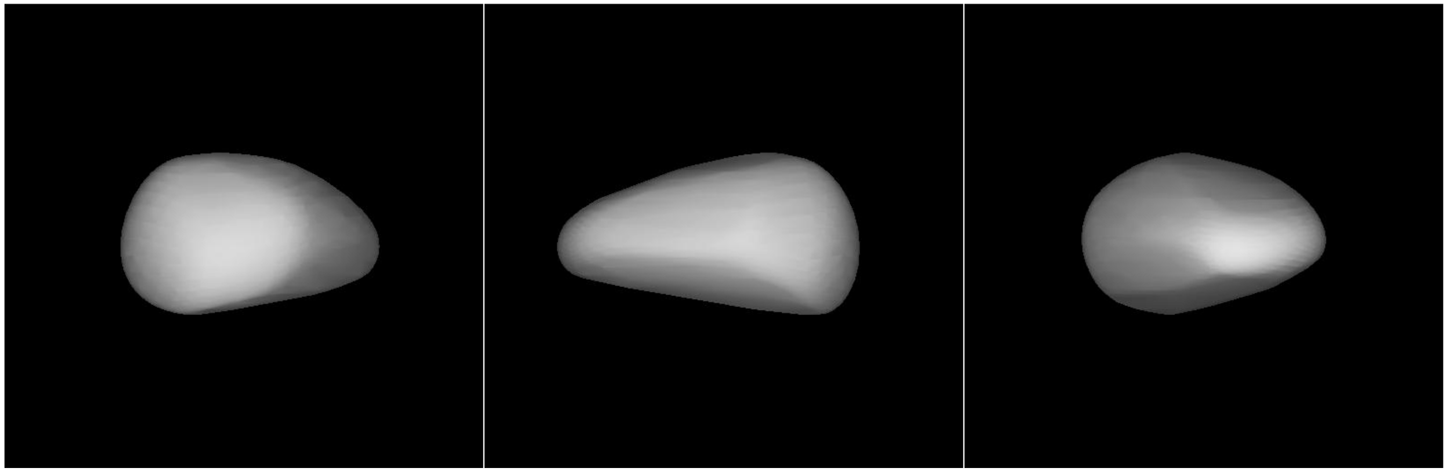
Inversion is not ill-posed, either!
Left: 2002 vertices, 4000 facets
Right: Laplace series, $l = 6$, $m = 6$ (258 vertices, 512 facets)

Asteroid examples:

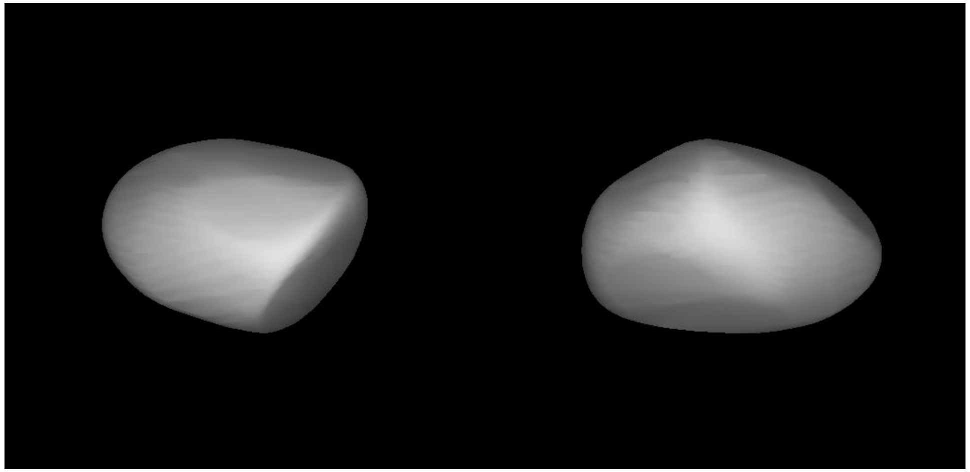
- Increasing irregularity as size decreases, but even large bodies may have irregular features



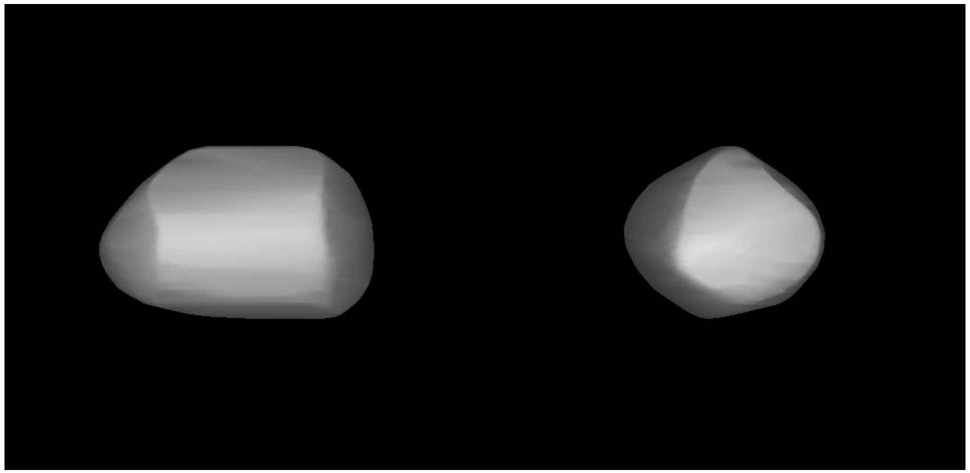
- Albedo variegation weak or moderate
- Possible large contact binaries (Nysa, Daphne, Hector)



- The smallest ones have widely varying shapes; often sharp, collision-fragment like features
- Despite the high solar phase angles, so far only implicit indications of large nonconvexities (scattering law not known accurately, so impossible to get nonconvex details → convex inversion converges better)

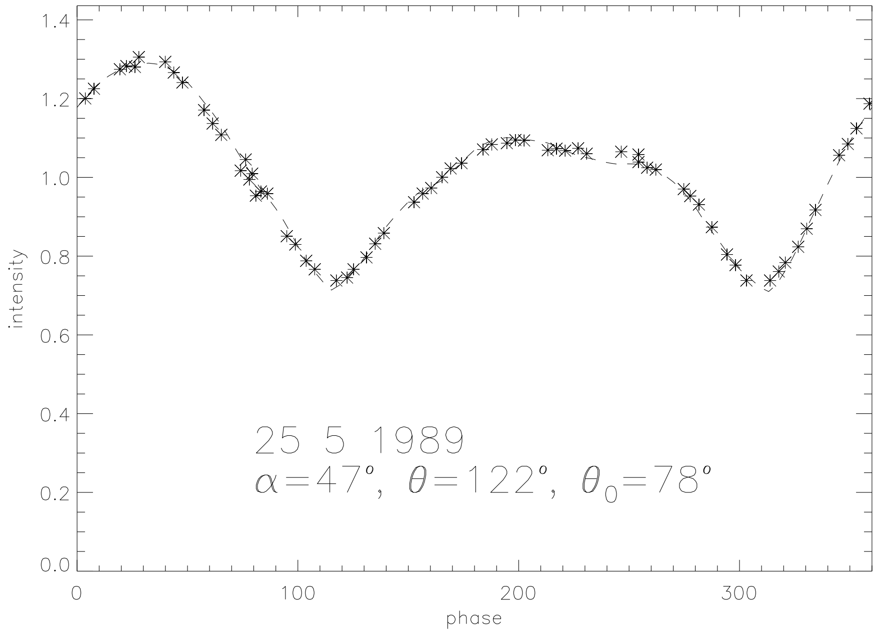


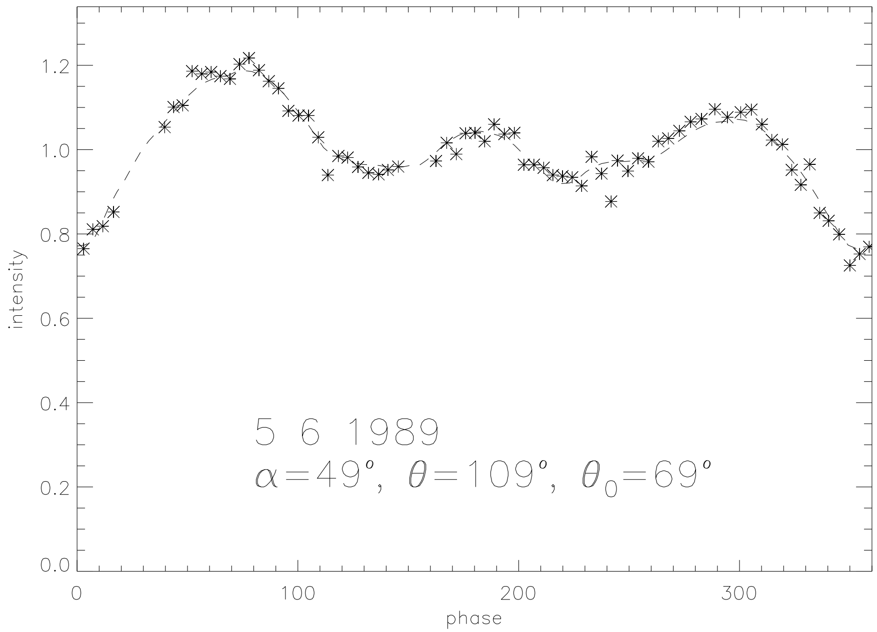
1580 Betulia



1980 Tezcatlipoca

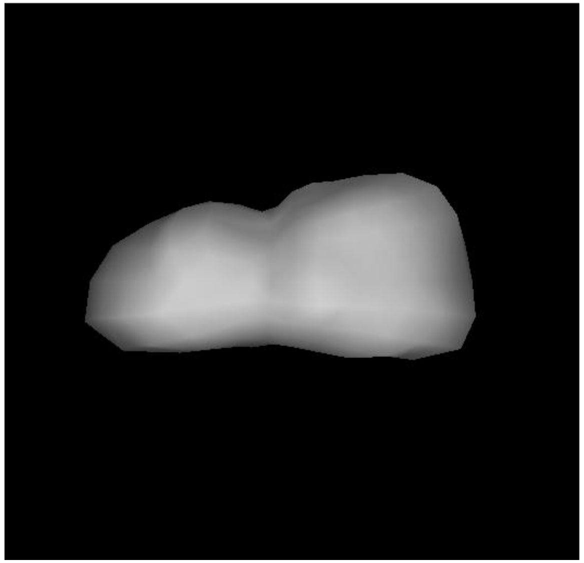
Lightcurves of 1580 Betulia:

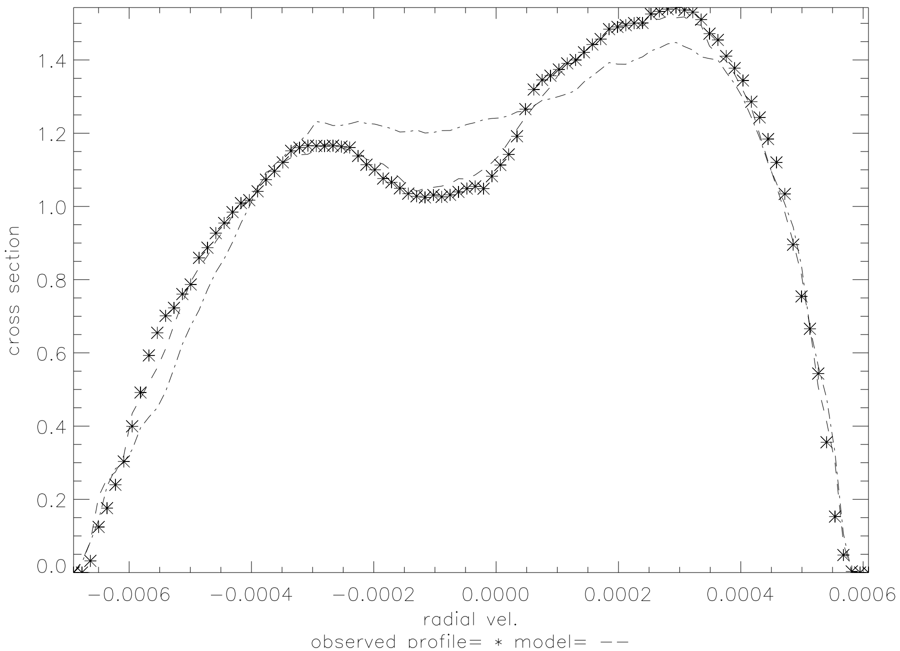




Multidatainversion with complementary data

- **Main sources of asteroid data:**
  1. Photometry (many targets, long range)
  2. *In situ* observations (very few targets)
  3. Delay-Doppler radar (relatively few targets, limited range)
  4. (Spectroscopy + astrometry)
- **Complementary data:**
  1. CW radar (Doppler only)
  2. Interferometry (Hubble Space Telescope/FGS, speckle)
  3. 'Snapshots': HST, adaptive optics
  4. Stellar occultations
  5. Thermal IR
  6. Ultraprecise astrometry/photocentre offset (GAIA etc.)
  7. Polarimetry
- Simultaneous optimization
  - $\chi^2_{\text{tot}} = \chi^2 + \sum_i \lambda_i \chi^2_i$
  - Rough data, small λ_i: constraints (binaries)
  - Accurate data, large λ_i: details (nonconvexities)




cross section
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.0006
−0.0004
−0.0002
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0.0002
0.0004
0.0006
radial vel.
observed profile= * model= --