Weights from parabolic equations

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References

- Olli Saari, *Parabolic BMO and global integrability of supersolutions to doubly nonlinear parabolic equations*, to appear in Rev. Mat. Iberoam.
- Juha Kinnunen and Olli Saari, *Parabolic weighted norm inequalities for partial differential equations*, available in arXiv.
- Juha Kinnunen and Olli Saari, *On weights satisfying parabolic Muckenhoupt conditions*, to appear in Nonlinear Anal.

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Plan of the talk

- **Goal:** theory of weights related to parabolic PDEs, generalization of one-sided weights to $\mathbb{R}^n$
- **Questions:** solutions as weights, parabolic BMO, applications to PDEs.
- **Tools:** techniques related to the weighted norm inequalities and one-sided weights, geometry of PDE.

Plan of the talk

(1) Muckenhoupt weights and elliptic equations
(2) Parabolic BMO and PDE
(3) One-sided and parabolic weights

Basic concepts I

- The Hardy-Littlewood maximal function of $f \in L^1_{loc}(\mathbb{R}^n)$ is

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f|,$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$ containing x.

- Let $w \in L^1_{loc}(\mathbb{R}^n)$, $w \geq 0$, be a weight. The Muckenhoupt A_p condition with $p > 1$ is

$$\sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{p-1} < \infty,$$

where $p' = p/(p-1)$.

Basic concepts II

- The Muckenhoupt A_1 condition is

$$\sup_Q \left(\fint_Q w \right) \left(\inf_Q w \right)^{-1} < \infty.$$

- The Muckenhoupt A_∞ class is $A_\infty = \bigcup_{p \geq 1} A_p$.
- Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $f \in \mathrm{BMO}$, if

$$\sup_Q \fint_Q |f - f_Q| < \infty.$$

Classics

The following statements are equivalent:

- $M : L^p(w) \to L^p(w)$, $p > 1$, is bounded,
- $w \in A_p$, (Muckenhoupt's theorem)
- $w = uv^{1-p}$ with $u, v \in A_1$. (Jones' factorization)

In addition

- $\mathrm{BMO} = \{\lambda \log w : w \in A_p, \lambda \in \mathbb{R}\}$, (John-Nirenberg lemma)
- $f \in \mathrm{BMO} \Leftrightarrow f = \alpha \log M\mu - \beta \log M\nu + b$ with μ, ν positive Borel measures with almost everywhere finite maximal functions, $b \in L^\infty(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$. (Coifman-Rochberg characterization)

Elliptic PDEs

A nonnegative weak solution $u \in W^{1,p}_{loc}(\Omega)$ to the elliptic p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad p \in (1, \infty),$$

satisfies the following properties:

- $\log u \in \mathrm{BMO}$, (logarithmic Caccioppoli's estimate)
- $u \in A_1$, (weak Harnack's inequality)
- $\sup_Q u \leq c \inf_Q u$. (Harnack's inequality)

An application

Theorem (Lindqvist 1993)

Let Ω be a "reasonable" domain. Then there is $\epsilon > 0$ such that if $u > 0$ is a supersolution in Ω, then

$$u^{\epsilon} \in L^1(\Omega).$$

Remark: This gives integrability up to the boundary whereas only local integrability was assumed a priori!

Q: Can we do similar things with parabolic equations?

A: Yes, in some cases.

The doubly nonlinear equation I

A weak solution to the doubly nonlinear equation

$$(|u|^{p-2}u)_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad p \in (1, \infty),$$

is a function $u \in L^p_{loc}(-\infty, \infty; W^{1,p}_{loc}(\mathbb{R}^n))$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(|\nabla u|^{p-2} \cdot \nabla \phi - |u|^{p-2} u \frac{\partial \phi}{\partial t} \right) \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\phi \in C_0^\infty(\mathbb{R}^{n+1})$.

The doubly nonlinear equation II

- The results also apply to equations with more general structure.
- $p = 2$ gives the heat equation.
- Uniformly parabolic linear equations will do for us as well.
- p-parabolic equation is not covered.

The Barenblatt solution

Example

The function

$$u(x,t) = ct^{\frac{-n}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}}, \, x \in \mathbb{R}^n, \, t > 0,$$

is a solution of the doubly nonlinear equation in the upper half space $\mathbb{R}^{n+1}_+$.

Observe: $u(x,t) > 0$ for every $x \in \mathbb{R}^n$ and $t > 0$. This indicates infinite speed of propagation of disturbances. When $p = 2$ we have the heat kernel.

Parabolic scaling

- If $u(x,t)$ is a solution, so is $u(\lambda x, \lambda^p t)$ with $\lambda > 0$.
- We call $(x,t) \mapsto (\lambda x, \lambda^p t)$ parabolic dilation/scaling.
- If $u(x,t)$ is a solution, so is $u((x,t)+z)$ with $z \in \mathbb{R}^{n+1}$.
- We will work in a geometry respecting these symmetries.

Parabolic rectangles

Definition

Let $Q = Q(x, l) \subset \mathbb{R}^n$ be a cube with center x and side length l. Let $\gamma \in [0, 1)$ and $t \in \mathbb{R}$. We denote

$$\begin{aligned} R = R(x, t, l) &= Q(x, l) \times (t - l^p, t + l^p), \\ R^+(\gamma) &= Q(x, l) \times (t + \gamma l^p, t + l^p) \quad \text{and} \\ R^-(\gamma) &= Q(x, l) \times (t - l^p, t - \gamma l^p). \end{aligned}$$

We say that R is a parabolic rectangle with center at (x, t) and sidelength l. $R^{\pm}(\gamma)$ are the upper and lower parts of R. Number γ is called the time lag.

Regularity theory I

- We have scale and location invariant Harnack's inequality

$$\sup_{R^-(\gamma)} u \leq C(n,p,\gamma) \inf_{R^+(\gamma)} u$$

  with $\gamma > 0$ for nonnegative weak solutions. (Moser 1964, Trudinger 1968, Kinnunen-Kuusi 2007)
- The time lag $\gamma > 0$ is an unavoidable feature of the theory rather than a mere technicality. This can be seen from the heat kernel already in the case $p = 2$.

Moser's proof of the parabolic Harnack inequality is based on

$$\sup_{R^-(\gamma)} u \lesssim \left(\fint_{(2R)^-(\gamma)} u^\varepsilon \right)^{1/\varepsilon},$$

$$\left(\fint_{(2R)^+(\gamma)} u^{-\varepsilon} \right)^{-1/\varepsilon} \lesssim \inf_{R^+(\gamma)} u$$

and on

$$\left(\fint_{(2R)^-(\gamma)} u^\varepsilon \right)^{1/\varepsilon} \lesssim \left(\fint_{(2R)^+(\gamma)} u^{-\varepsilon} \right)^{-1/\varepsilon}.$$

The last step is proved as follows

- $-\log u$ is in "parabolic BMO"
- A parabolic John–Nirenberg lemma holds
- u satisfies the "parabolic A_2 condition".

Parabolic BMO

Definition

Let $f \in L^1_{loc}(\Omega_T)$ and $\gamma \in (0,1)$. We say that $f \in \mathrm{PBMO}^+(\Omega_T)$ if for each parabolic rectangle R there is a constant a_R such that

$$\sup_R \left(\fint_{R^+(\gamma)} (f - a_R)_+ + \fint_{R^-(\gamma)} (f - a_R)_- \right) < \infty,$$

where the supremum is taken over all parabolic rectangles $2R \subset \Omega_T$. If the condition above is satisfied with the direction of the time axis reversed, we denote $f \in \mathrm{PBMO}^-$.

Remarks

- Original condition in papers by Moser, Fabes and Garofalo was
$$\sup_R \left(\fint_{R(0)^+} \sqrt{(f-a_R)_+} + \fint_{R(0)^-} \sqrt{(f-a_R)_-} \right) < \infty.$$
These functions are included in our PBMO$^+$.
- Even if we begin with a definition without lag, the lag appears in the John-Nirenberg lemma: Let $u \in \text{PBMO}^+$, $\gamma \in (0,1)$, R a parabolic rectangle. Then for $A, B \eqsim_{n,p,u} 1$ we have
$$|R^+(\gamma) \cap \{(u-a_R)_+ > \lambda\}| \leq Ae^{-B\lambda}|R^+(\gamma)|$$
and
$$|R^-(\gamma) \cap \{(u-a_R)_- > \lambda\}| \leq Ae^{-B\lambda}|R^-(\gamma)|.$$
(Moser, Aimar, Garofalo-Fabes)
- The lag $\gamma > 0$ in the definition allows us to characterize PBMO$^+$ with a John–Nirenberg inequality. The John-Nirenberg inequality cannot hold with $\gamma = 0$.

Global John–Nirenberg

Theorem (S. 2014)

Let $u \in \mathrm{PBMO}^+(\Omega_T)$*,* $\delta > 0$ *and* Ω *be a nice domain (John will do). There is* $c \in \mathbb{R}$ *and* $A, B \eqsim_{n,p,u,T,\delta,\Omega} 1$ *such that*

$$|(\Omega_T \setminus \Omega_\delta) \cap \{(u-c)_+ > \lambda\}| \le Ae^{-B\lambda}|\Omega_T \setminus \Omega_\delta|$$

for all $\lambda > 0$*.*

A consequence

Corollary (S. 2014)

Let u be a positive lower semicontinuous supersolution to the doubly nonlinear equation on $\Omega \times (0, T)$, where Ω is a nice (John, for instance) domain. Then there exists $\epsilon > 0$ such that for all $\delta > 0$

$$u^{\epsilon} \in L^1(\Omega \times (0, T - \delta)).$$

Remark: This result seems to be new even for the heat equation.

Parabolic Muckenhoupt condition I

We have defined a class of functions deserving the name parabolic BMO.

Next we will turn to corresponding Muckenhoupt conditions. **Why?**

We want

- to get Coifman-Rochberg (maximal function) characterization for PBMO^+
- to give a nice generalisation of one-sided weights to $\mathbb{R}^n$, $n \geq 2$.

Parabolic Muckenhoupt condition II

- Consider the *one-sided* maximal function:

$$M^{+}f(t) = \sup_{h>0} \frac{1}{h} \int_{t}^{t+h} |f| \, \mathrm{d}t.$$

- Sawyer characterized its strong and weak type weighted norm inequalities through

$$\sup_{x,h} \frac{1}{h} \int_{x-h}^{x} w \left(\frac{1}{h} \int_{x}^{x+h} w^{1-q'} \right)^{q-1} < \infty.$$

- This *one-sided* condition creates a complete analogue of A_p theory on real line.

Parabolic Muckenhoupt condition III

- The same problem with two or more variables has turned out to be quite difficult. Some partial results are known (Berkovits, Forzani, Lerner, Martín-Reyes, Ombrosi 2010–2011). It is not known if they can be improved.
- We suggest a very different approach to this problem.

Parabolic Muckenhoupt condition IV

Definition

Let $\gamma \in (0,1)$ and $q > 1$. $w \in L^1_{loc}(\mathbb{R}^{n+1})$, $w > 0$, is in the parabolic Muckenhoupt class $A_q^+(\gamma)$, if

$$\sup_R \left(\fint_{R(\gamma)^-} w \right) \left(\fint_{R(\gamma)^+} w^{1-q'} \right)^{q-1} < \infty,$$

where the supremum is over all parabolic rectangles $R \subset \mathbb{R}^{n+1}$. If the condition above is satisfied with the direction of the time axis reversed, we denote $w \in A_q^-(\gamma)$.

Observe: The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the rectangles $R^-(\gamma)$ and $R^+(\gamma)$ is essential for us.

Remarks

- Classical A_q weights with a trivial extension in time belong to the parabolic $A_q^+(\gamma)$ class.
- Moreover, if $w \in A_q^+(\gamma)$, then $e^t w \in A_q^+(\gamma)$.
- Parabolic $A_q^+(\gamma)$ weights are not necessarily doubling.

Elementary properties

- $1 < q < r < \infty \Rightarrow A_q^+(\gamma) \subset A_r^+(\gamma)$. (Inclusion)
- $w \in A_q^+(\gamma) \Leftrightarrow w^{1-q'} \in A_{q'}^-(\gamma)$. (Duality)
- Let $w \in A_q^+(\gamma)$ and $E \subset R^+(\gamma)$. Then

$$w(R^-(\gamma)) \leq C \left(\frac{|R^-(\gamma)|}{|E|} \right)^q w(E).$$

(Forward in time doubling)

Lemma (Kinnunen-S. 2014)

If $w \in A_q^+(\gamma)$ *for some* $\gamma \in [0,1)$*, then* $w \in A_q^+(\gamma')$ *for all* $\gamma' \in (0,1)$.

The parabolic maximal operator

Definition

Let $f \in L^1_{loc}(\mathbb{R}^{n+1})$ and $\gamma \in (0,1)$. We define the parabolic forward in time maximal function

$$M^{\gamma+} f(x,t) = \sup \fint_{R^+(\gamma)} |f|,$$

where the supremum is taken over all parabolic rectangles $R(x,t)$ centered at (x,t). The parabolic backward in time operator $M^{\gamma-}$ is defined analogously.

Observe: The definition makes sense also for $\gamma = 0$, but the lag $\gamma > 0$ between the point (x,t) and the rectangle $R^+(\gamma)$ is essential for us.

Characterization of parabolic Muckenhoupt weights

Theorem (Kinnunen-S. 2014)

Let $q > 1$. The following claims are equivalent:

- $w \in A_q^+(\gamma)$ *for some* $\gamma \in (0,1)$,
- $w \in A_q^+(\gamma)$ *for all* $\gamma \in (0,1)$,
- $M^{\gamma+} : L^q(w) \to L^q(w)$ *for all* $\gamma \in (0,1)$, *(the strong type estimate)*
- $M^{\gamma+} : L^q(w) \to L^{q,\infty}(w)$ *for some* $\gamma \in (0,1)$. *(the weak type estimate)*

Proof.

- The parabolic Muckenhoupt $A_q^+(\gamma)$ conditions are equivalent for all $\gamma \in (0,1)$. This is needed to prove that the $A_q^+(\gamma)$ condition is necessary.
- The sufficiency part of the weak type estimate uses a modification (parabolic rectangles $n \geq 2$) of a covering argument by Forzani, Martín-Reyes and Ombrosi.
- The strong type estimate follows from a reverse Hölder type inequality, the equivalence of $A_q^+(\gamma)$ conditions and interpolation.

$\square$

Reverse Hölder inequality

Lemma (Kinnunen-S. 2014)

Let $w \in A_q^+(\gamma)$ and $\gamma \in (0,1)$. Then there is $\varepsilon > 0$ such that

$$\left(\fint_{R^-(0)} w^{1+\varepsilon} \right)^{1/(1+\varepsilon)} \leq C \fint_{R^+(0)} w$$

for every parabolic rectangle $R \subset \mathbb{R}^{n+1}$.

Observe: This is weaker than the standard RHI, because the rectangles $R^-(0)$ and $R^+(0)$ are not the same. Otherwise, we would have the standard A_∞ condition.

Proof.

- First we prove a distribution set estimate

$$w(\widehat{R} \cap \{w > \lambda\}) \leq C\lambda|\widetilde{R} \cap \{w > \beta\lambda\}|,$$

  where $\widehat{R}$ and $\widetilde{R}$ are certain parabolic recangles.
- Some care must be taken, since there are no obvious dyadic cubes.
- Once the distribution set estimate is done, the claim follows (quite) easily.

$\square$

A Coifman-Rochberg type result

Corollary (Kinnunen-S. 2014)

Let $f \in \mathrm{PBMO}^+$ and $0 < \gamma < 1$. Then there are positive Borel measures μ, ν satisfying

$$M^{\gamma+}\mu < \infty \quad \text{and} \quad M^{\gamma-}\nu < \infty$$

almost everywhere in $\mathbb{R}^{n+1}$, a bounded function b and constants $\alpha, \beta \geq 0$ such that

$$f = -\alpha \log M^{\gamma+}\mu + \beta \log M^{\gamma-}\nu + b.$$

Conversely, if the above holds with $\gamma = 0$, then $f \in \mathrm{PBMO}^+$.

An open question

Question: Does

$$\sup_R \left(\fint_{R^-(\gamma)} w \right) \exp \left(\fint_{R^+(\gamma)} \log w^{-1} \right) < \infty$$

imply that $w \in A_q^+(\gamma)$ for some q?

Reformulation: We say that $u \in \mathrm{BMO}^+$ if

$$\sup_R \fint_{R^-(\gamma)} (u - u_{R^+(\gamma)})_+ < \infty.$$

Does this imply $u \in \mathrm{PBMO}^-$? For a positive answer, it would suffice to show

$$\sup_R \fint_{R^+(\gamma)} (u_{R^-(\gamma)} - u)_+ < \infty.$$