Exceptional sets for the p-parabolic equation

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The Laplace equation

Consider the following problem:

- Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a smooth and bounded domain.
- Let $K \subset \Omega$ be compact.
- Consider the functions $u \in C_0^\infty(\Omega)$ with $u \geq 1_K$.

What is the minimal value

$$\mathrm{cap}_2(K) := \inf_u \int_\Omega |\nabla u|^2$$

and which function is the extremizer?

The Laplace equation

- Without the *constraint* $u \geq 1_K$, the minimizing function is clearly 0.
- In general, it is an easy consequence of a classical fact that a necessary condition for being a minimizer is that

$$0 = \frac{d}{d\epsilon} \left(\int_\Omega |\nabla(u + \epsilon\varphi)|^2 \right)_{\epsilon=0} = 2 \int_\Omega \nabla u \cdot \nabla \varphi = -2\langle \Delta u, \varphi \rangle$$

  for all $\varphi \in C_0^\infty(\Omega \setminus K)$.
- u is a point in a vector space, φ is a search direction, and Δu is a generalized "gradient" of the functional.
- If u is the solution to the constrained problem, the feasible directions increase the value of the functional.
- So $-\Delta u$ should be a positive distribution supported in K.

Terminology

- u as before is an example of *superharmonic* function. A solution to the Laplace equation is *harmonic*.
- If μ is a positive Radon measure and $-\Delta u = \mu$, then we call u the potential of μ. We call μ the source (or Riesz measure) of u.
- Actually all superharmonic functions can be identified with solutions to measure data problems.
- It depends on μ which function space u can belong to.
- The case $u \in W^{1,2}$ and $\mu \in (W^{1,2})'$ is natural. (supersolution)
- The case with a Radon measure μ and $u \notin W^{1,2}$ is interesting too. (superharmonic)

$u \notin W^{1,2}(\Omega)$?

- Functions in this class should not arise in the minimization process.
- Corresponding Riesz measures shouldn't do it either.

When does 1_K behave like 0 in $W^{1,2}$?
What are the sets where superharmonic functions are really bad?
How small sets K the Laplacian does not see from its natural domain?

The classical capacity

The following are equivalent

- K has capacity zero.
- All bounded and harmonic u on $\Omega \setminus K$ can be continued harmonically onto whole of Ω.
- There is a superharmonic u with $u|_K = \infty$.
- $W^{1,2}(\Omega) \simeq W^{1,2}(\Omega \setminus K)$.
- No Radon measure μ supported in K has bounded potential.

The p-parabolic equation

- In this talk, we will study the nonlinear and evolutionary version of the problem.
- We will study the variational integral

$$\int_{\Omega} |\nabla u|^p$$

with $p > 2$. The corresponding Euler-Lagrange equation is the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

- And finally, the corresponding equation for "cooling down" is

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

The parabolic Sobolev space

The natural space for the solutions is the parabolic Sobolev space: $u \in L^p(0,T;W_0^{1,p}(\Omega))$ if

- $u(\cdot,t) \in W_0^{1,p}(\Omega)$ for a.e. $t \in (0,T)$.
- $t \mapsto \|u(\cdot,t)\|_{W^{1,p}(\Omega)}$ is measurable.
- $\int_0^T \int_\Omega |\nabla u|^p < \infty$.

Note that the gradient is only with respect to the space variables! We will denote

$$\mathcal{V} = L^p(0,T;W_0^{1,p}(\Omega)).$$

The solutions

We call $u \in \mathcal{V}_{loc}$ a solution if

$$\int_0^T \int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \partial_t \varphi) \, \mathrm{d}x \, \mathrm{d}t = 0$$

for all $\varphi \in C_0^\infty$.

For a Radon measure μ, we define the solution to measure data problem

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \mu$$

through a similar formula. (mind the function space!)

Supersolutions

There are three types of supersolutions:

1. Supersolutions. The weak formulation presented previously with ≥ 0 and non-negative test functions φ. They are in the natural space $\mathcal{V}$.
2. Good superparabolic functions. Solutions to measure data problems without requirement on the membership to $\mathcal{V}$.
3. Bad superparabolic functions. They are defined through comparison principle, lower semi-continuity and finiteness in a dense set, and they do not solve any measure data problem.

In our considerations, we will not deal with the third class. It is empty in the linear theory.

Features of the p-parabolic setting

- Disturbances in initial data propagate at finite speed.
- Sums of solutions and scaled solutions need not be solutions to the p-parabolic equation.
- There is no a priori assumption on the smoothness of the time derivative of the solution.

Capacity

We want to measure the sets that are negligible in the study of the p-parabolic equation. There are several restrictions.

- Unlike the (p-)Laplace equation, we do not have a(n obvious) function space norm to minimize.
- Unlike the heat equation, we do not have a representation formula (heat kernel) for the solution.
- A natural capacity should coincide with the known ones as we let $p = 2$ or freeze the time variable.

In what follows we go on with using the terminology of potentials even if the equations is different to the one in the introduction.

The nonlinear parabolic capacity [KKKP]

The nonlinear parabolic capacity of a set $E \subset \Omega_\infty$ is defined by

$$\mathrm{cap}(E) = \sup\{\mu(\mathbb{R}^{n+1}) : \operatorname{supp}\mu \subset E, 0 \leq u_\mu \leq 1\}.$$

Here u_μ is the solution to the measure data problem of μ with zero boundary values on the parabolic boundary. It will also be called the potential of μ. The supremum is taken over all positive Radon measures on $\mathbb{R}^{n+1}$.

The definition is motivated by the dual formulation of the capacity for the Laplace equation.

Remarks

- The definition of the capacity is tightly connected to the equation instead of coming from a function space.
- Contrary to the linear case, there is no representation formula for the potential (at the moment). This makes the definition difficult to use.
- There is, however, an easier way to compute capacities: the variational capacity.

Variational capacity

Recall the space $\mathcal{V} = L^p(0,\infty; W_0^{1,p}(\Omega))$. Let

$$\|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}}^p + \|u_t\|_{\mathcal{V}'}^{p'}.$$

Define the *variational capacity* of a compact set $K \subset \Omega_\infty$ by

$$\mathrm{cap}_{var}(K) = \inf\{\|\varphi\|_{\mathcal{W}} : \varphi \in C_0^\infty(\Omega \times \mathbb{R}), \varphi \geq 1_K\}.$$

The definition is extended to Borel sets in the usual way by approximating by open sets from outside and compact sets from inside.

Variational capacity

Theorem (Avelin-Kuusi-Parviainen 2015)

Let K be a compact set. Then

$$\mathrm{cap}_{var}(K) \approx \mathrm{cap}(K).$$

Exceptional sets

Theorem (Avelin-S 2015)

Let K be compact. Then the following are equivalent.

- $\mathrm{cap}(K) = 0$.
- *Bounded solutions and supersolutions to the p-parabolic equation can be extended over K.*
- *There is a p-superparabolic function with $u|_K = \infty$.*

Hausdorff measure

- Consider the translation invariant metric on $\mathbb{R}^{n+1}$:

$$d((x,t),(0,0)) := \max\{|x|, |t|^{1/p}\}.$$

- Let $\mathcal{P}^s$ with $s \in (0,\infty)$ be the corresponding Hausdorff measure.
- Restricted to lateral hyperplanes $\mathbb{R}^n \times \{t_0\}$, it becomes the n-dimensional Lebesgue measure.
- It gives dimension p to the time axis.
- More complicated sets may be more complicated.

Hausdorff measure

Theorem (Avelin-S 2015)

Let A be a Borel set.

- *If* $\mathcal{P}^n(A) = 0$, *then* $\operatorname{cap}(A) = 0$.
- *If* $\operatorname{cap}(A) = 0$, *then* $\mathcal{P}^s(A) = 0$ *for all* $s > n$.

Lemmas for proving Hausdorff

Lemma (Frostman's lemma (Howroyd))

Let (X, d) be a metric space, and let $A \subset X$ be a Borel set. If $\mathcal{H}^s(A) > 0$, then there is a Radon measure μ supported in A such that

$$\mu(E) \leq d(E)^s$$

for all $E \subset X$.

Lemmas for proving Hausdorff

Lemma (Avelin-Kuusi-Mingione, 2015)

Let u be a weak solution of the measure data problem

$$u_t - \Delta_p u = \mu$$

with finite Radon measure and zero boundary and initial values in Ω_∞. If

$$\left[\int_0^r \left(\frac{\mu(Q_\rho^-(x,t))}{\rho^n}\right)^{\frac{p}{n(p-2)+p}} \frac{\mathrm{d}\rho}{\rho}\right]^{\frac{n(p-2)+p}{p}}$$

is locally bounded for some r, where Q_ρ^- is the lower half of $Q_\rho = B(x,\rho) \times (t-\rho^p, t+\rho^p)$, then u is locally bounded.

Proof of the Hausdorff measure estimates

- By subadditivity of the capacity together with an estimate for the capacity of parabolic cylinders it is clear that $\mathrm{cap}(A) \lesssim_{p,n} \mathcal{P}^n(A)$.
- The other direction is proved by a contradiction. If $\mathcal{P}^n(A) > 0$, then Frostman and potential estimate imply that there exists an admissible superparabolic function whose source charges A.

- Infinity sets of superparabolic functions are called polar.
- It was already proved in [KKKP] that polar sets have zero capacity.
- We will next discuss the proof of the converse statement. Along the way the tools that were needed for the first part will also be introduced.

The obstacle problem

Let $\psi \in C_0(\Omega_\infty)$ and

$$S_\psi = \{u : u \geq \psi, u \text{ is superparabolic}\}.$$

We let

$$R^\psi(z) = \inf_{u \in S_\psi} u(z)$$

$$\widehat{R}^\psi(z) = \lim_{r \to 0} \inf_{x \in B(z,r)} R^\psi(x)$$

We call $\widehat{R}^\psi(z)$ the solution to the obstacle problem.

The obstacle problem

- Solution to the obstacle problem is superparabolic.
- If $\psi_1 \leq \psi_2$ then the solutions to corresponding obstacle problems preserve the order. [LP]
- Characteristic functions of compact sets can be used as obstacles after an approximation procedure.
- In this case, the solution to the obstacle problem $u = \widehat{R}^K$ is *the capacitary potential of K*:

$$u_t - \Delta_p u = \mu$$

  where the data is the *capacitary distribution* $\mu(\mathbb{R}^{n+1}) = \text{cap}(K)$. [KKKP]
- The capacitary potential is a quasiminimizer of the energy $\int_{\Omega_\infty} |\nabla u|^p$ among superparabolic functions with the correct boundary values. [AKP]

The obstacle problem

Lemma (Avelin-Kuusi-Parviainen)

Let $\psi \in C_0^\infty(\Omega \times \mathbb{R})$. If u solves the obstacle problem in Ω_∞ with ψ as the obstacle, that is, $u = \hat{R}^\psi$, then $u \geq \psi$ is a continuous supersolution and

$$\|u\|_{en} \lesssim \|\psi\|_{\mathcal{W}(\Omega_\infty)}.$$

Construction of an infinity

- Assume that $K \subset \Omega_\infty$ has capacity zero. We will construct a superparabolic function that is infinite in K.
- Since $\mathrm{cap}_{var}(K) = 0$, we may find $\varphi_j \in C_0^\infty(\Omega_\infty)$ such that

$$\|\varphi_j\|_{\mathcal{W}} \leq 2^{-(p+1)j}$$

  and $\varphi_j \geq 1_K$.
- Partial sums of $\sum_j \varphi_j$ have uniformly bounded (quasi)norm in $\mathcal{W}$.
- Using them as obstacles, we obtain an increasing sequence of superparabolic functions with bounded *energy*.
- The increasing limit can be shown to be a supersolution function. It is clear that it is infinite in K.

Thank you for your attention!