Parabolic BMO and the forward-in-time maximal operator

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10th International Conference on Harmonic Analysis and Partial Differential Equations
El Escorial, Madrid

June 15, 2016

BMO

- A function $u \in L^1_{loc}$ is said to be of *bounded mean oscillation* ($u \in \text{BMO}$) if

$$\|u\|_{\text{BMO}} = \sup_Q \fint_Q |u - u_Q| < \infty.$$

- The remarkable John-Nirenberg inequality asserts that

$$\sup_Q \fint_Q \exp(\epsilon |u - u_Q|) < \infty$$

for some positive $\epsilon \lesssim \|u\|_{\text{BMO}}^{-1}$.

- BMO is connected to many questions of harmonic analysis. In particular, $\text{BMO} = \{\alpha \log w : w \in A_2, \alpha \in \mathbb{R}\}$.
- There is also an interesting connection between BMO, A_2 and the regularity theory of elliptic PDE of divergence form.

Elliptic PDE

- Let A be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi \cdot A\xi \leq \Lambda|\xi|^2$$

for some $\Lambda \in (1,\infty)$ uniformly in x.

- If w is a positive weak (super)solution to

$$\mathrm{div}(A\nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \mathrm{BMO}(\Omega)$. This is an important observation in Moser's proof of the DeGiorgi–Nash–Moser theorem.

- As a consequence, $w^\epsilon \in A_2$. It is also true that $w \in A_1$.
- Recall that $w \in A_p$ if

$$[w]_{A^p} = \sup_{Q\subset\Omega} \fint_Q w \left(\fint_Q w^{1-p'}\right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$

Parabolic PDE

- BMO arises in an intrinsic manner from elliptic PDE. We would like to see what happens with parabolic ones.
- We consider local solutions to e.g. one of the following

$$
\begin{aligned}
&u_t - \Delta u = 0, \\
&u_t - \operatorname{div}(A\nabla u) = 0, \\
&(u^{p-1})_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0
\end{aligned}
$$

  in $\Omega \times (0, T)$. For our purposes, the last one is the most general one, and we will concentrate on it.
- In general, the positive solutions cannot be Muckenhoupt A_2 weights in any obvious way (they can fail to be doubling measures with respect to any reasonable metric). Consequently, parabolic BMO must encode this "non-doubling" feature.

Contents of the talk

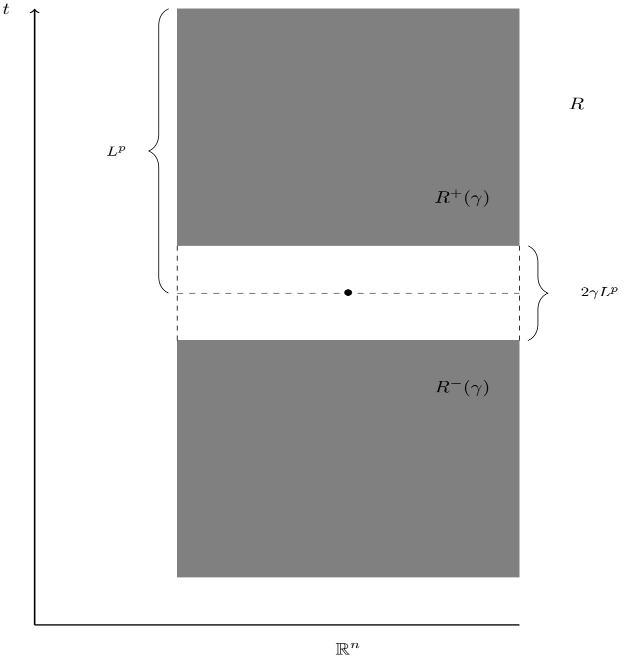
I will give a summary of the recent results about parabolic BMO arising from PDE. A part of the work is joint with J. Kinnunen. The rest of the talk consists of

1. Notation in the space time $\mathbb{R}^{n+1}$.
2. The definition of parabolic BMO.
3. Weights.
4. The forward-in-time maximal operator.

Notation

- The basic structure of $u_t - \Delta u = 0$ and its generalizations is preserved under translations $z \mapsto z + h$ and anisotropic dilations $(x, t) \mapsto (\delta x, \delta^p t)$ of the coordinates. ($p = 2$ for the heat equation)
- These transformations generate parabolic rectangles. We denote

$$\begin{aligned} R = R(x,t,L) &= Q(x,L) \times (t - L^p, t + L^p), \\ R^+(\gamma) &= Q(x,L) \times (t + \gamma L^p, t + L^p) \quad \text{and} \\ R^-(\gamma) &= Q(x,L) \times (t - L^p, t - \gamma L^p). \end{aligned}$$



t
L^p
R
$R^+(\gamma)$
$2\gamma L^p$
$R^-(\gamma)$
$\mathbb{R}^n$

The starting point in PDE

- It was discovered in the 1960s that the solutions to parabolic equations f satisfy

$$\fint_{R^+(0)} \fint_{R^-(0)} \sqrt{(u(x) - u(y))^+} \,\mathrm{d}x \,\mathrm{d}y < C(n, p)$$

for $u = -\log f$. (Moser, Trudinger)

- The parabolic John-Nirenberg lemma (Moser, Trudinger, Aimar) tells that

$$\fint_{R^+(\gamma)} \fint_{R^-(\gamma)} \exp(\epsilon(u(x) - u(y))^+) \,\mathrm{d}x \,\mathrm{d}y < C(n, p, \gamma)$$

for any $\gamma \in (0, 1)$.

The definition of PBMO$^-$, S. 2014

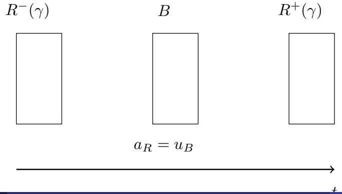
- **Definition**: $u \in \mathrm{PBMO}^-$ if

$$\|u\|_{\mathrm{PBMO}^-} := \sup_{R} \inf_{a} \left(\fint_{R^-(\frac{1}{2})} (u-a)^+ + \fint_{R^+(\frac{1}{2})} (a-u)^+ \right) < \infty.$$

- **Theorem**: It holds that

$$\|u\|_{\mathrm{PBMO}^-} \eqsim_{n,p,\gamma} \sup_{R} \inf_{a} \left(\fint_{R^-(\gamma)} (u-a)^+ + \fint_{R^+(\gamma)} (a-u)^+ \right).$$

- **Corollary**: It is possible to replace the constant a of the definition by the mean value in a certain cylinder:



Relation to standard BMO

- Recall the (modified) definition of the standard BMO: $u \in \text{BMO}$ if $u \in L^1_{loc}$ and

$$\sup_R \fint_R |u - u_R| < \infty,$$

  the supremum being taken over all parabolic rectangles.
- We have $\text{BMO} = \text{PBMO}^- \cap \text{PBMO}^+$ and none of the three classes of functions coincide.

Weights I

- The weights $A_q^+(\gamma)$ corresponding to PBMO$^-$ are the ones satisfying

$$\sup_R \fint_{R^-(\gamma)} w \left(\fint_{R^+(\gamma)} w^{1-q'} \right)^{q-1} < \infty, \quad 1 < q < \infty.$$

- As in the case of PBMO$^-$, we have that $A_q^+(\gamma) = A_q^+(\gamma')$ for all $\gamma, \gamma' \in (0,1)$.
- It holds

$$\text{PBMO}^- = \{\alpha \log w : w \in A_q^+, \alpha \in (0,\infty), q \in (1,\infty)\}.$$

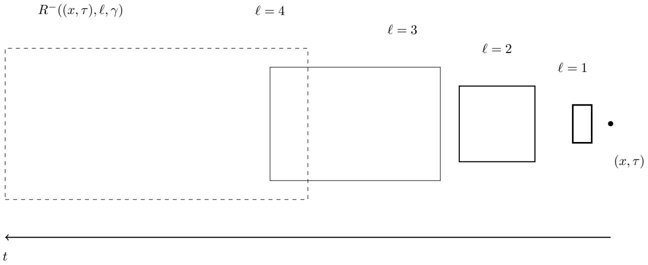
(Kinnunen and S. 2014)

Weights II

- We define the forward-in-time maximal function as

$$M^{\gamma+} f(z) := \sup_{\ell>0} \fint_{R^+(z,\ell,\gamma)} |f|.$$

- For $q \in (1, \infty)$, the operator $M^{\gamma+} : L^q(w) \to L^q(w)$ is bounded if and only if $w \in A_q^+(\gamma)$ (Kinnunen and S. 2014).



PBMO and forward-in-time maximal function, S. 2016

Theorem

Let $u \in \mathrm{PBMO}^+$ *be non-negative. If* $M^{\gamma+}u \in L^1_{loc}$*, then* $M^{\gamma+}u \in \mathrm{PBMO}^+$.

- The theorem holds true in $\mathbb{R}^{n+1}$ and $\Omega \times \mathbb{R}$.
- Nonnegativity is necessary at least in the latter case.